1. (a)

$$\phi(x) = v + \frac{1}{\sqrt{2}}\left(\psi(x) + i\chi(x)\right)$$

$$\mathcal{H} = \partial_t\phi^*\partial_t\phi + \nabla\phi^*\cdot\nabla\phi + \frac{\lambda}{4}\left(\phi^*\phi - v^2\right)^2$$

$$= \frac{1}{2}\left(\partial_t\psi(x) - i\,\partial_t\chi(x)\right)\left(\partial_t\psi(x) + i\,\partial_t\chi(x)\right)$$

$$+ \frac{1}{2}\left(\partial_j\psi(x) - i\partial_j\chi(x)\right)\left(\partial_j\psi(x) + i\partial_j\chi(x)\right)$$

$$+ \frac{\lambda}{4}\Big(\big(v + \frac{1}{\sqrt{2}}(\psi(x) - i\chi(x))\big)\big(v + \frac{1}{\sqrt{2}}(\psi(x) + i\chi(x)\big) - v^2\Big)^2$$

$$= \frac{1}{2}(\partial_t\psi)^2 + \frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}(\partial_t\chi)^2 + \frac{1}{2}(\nabla\chi)^2$$

$$+ \frac{\lambda}{4}\Big(\cancel{v^2} + \frac{1}{2}(\psi^2+\chi^2) + \frac{v}{\sqrt{2}}(\psi + \cancel{i\chi} + \psi - \cancel{i\chi}) \cancel{- v^2}\Big)^2$$

$$= \frac{1}{2}(\partial_t\psi)^2 + \frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}(\partial_t\chi)^2 + \frac{1}{2}(\nabla\chi)^2$$

$$+ \frac{\lambda}{4}\Big(\frac{1}{4}(\psi^4 + \chi^4 + 2\chi^2\psi^2) + 2v^2\psi^2 + \sqrt{2}\,v(\psi^3 + \chi^2\psi)\Big)$$

$$\boxed{\begin{aligned}\mathcal{H}_0 &= \frac{1}{2}(\partial_t\psi)^2 + \frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}(\partial_t\chi)^2 + \frac{1}{2}(\nabla\chi)^2 + \frac{\lambda}{2}v^2\psi^2\\ \mathcal{H}_{int} &= \frac{\lambda}{16}(\psi^4 + \chi^4 + 2\chi^2\psi^2) + \frac{\lambda v\sqrt{2}}{4}(\psi^3 + \chi^2\psi)\end{aligned}}$$

(b) The fields can be expanded as

$$\psi(x,t) = \int\frac{d^3k}{\sqrt{(2\pi)^3 2\omega_k}}\left(a_{\vec k}\, e^{-i\omega_{\vec k}t + i\vec k\vec x} + a^*_{\vec k}\, e^{i\omega_{\vec k}t - i\vec k\vec x}\right)$$

$$\chi(x,t) = \int\frac{d^3k}{\sqrt{(2\pi)^3 2\,\varepsilon_{\vec k}}}\left(b_{\vec k}\, e^{-i\varepsilon_{\vec k}t + i\vec k\cdot\vec x} + b^*_{\vec k}\, e^{i\varepsilon_{\vec k}t - i\vec k\cdot\vec x}\right)$$

. The part of $\mathcal{H}_0$ involving only ψ fields is

$$\int d^3x\,\mathcal{H}_0^\psi = \int d^3x\int\frac{d^3k\,d^3q}{(2\pi)^3 2\sqrt{\omega_{\vec k}\omega_{\vec q}}}$$

$$\Big(-\frac{\omega_{\vec k}\omega_{\vec q}}{2}\left(a_{\vec k}e^{-i\omega_{\vec k}t + i\vec k\cdot\vec x} - a^*_{\vec k}e^{i\omega_{\vec k}t - i\vec k\cdot\vec x}\right)\left(a_{\vec q}e^{-i\omega_{\vec q}t + i\vec q\cdot\vec x} - a^*_{\vec q}e^{i\omega_{\vec q}t - i\vec q\cdot\vec x}\right)$$

$$- \frac{\vec k\cdot\vec q}{2}\left(a_{\vec k}e^{-i\omega_{\vec k}t + i\vec k\cdot\vec x} - a^*_{\vec k}e^{i\omega_{\vec k}t - i\vec k\cdot\vec x}\right)\left(a_{\vec q}e^{-i\omega_{\vec q}t + i\vec q\cdot\vec x} - a^*_{\vec q}e^{i\omega_{\vec q}t - i\vec q\cdot\vec x}\right)$$

$$+ \frac{\lambda v^2}{2}\left(a_{\vec k}e^{-i\omega_{\vec k}t + i\vec k\cdot\vec x} + a^*_{\vec k}e^{i\omega_{\vec k}t - i\vec k\cdot\vec x}\right)\left(a_{\vec q}e^{-i\omega_{\vec q}t + i\vec q\cdot\vec x} + a^*_{\vec q}e^{i\omega_{\vec q}t - i\vec q\cdot\vec x}\right)\Big)$$

integrating over x gives momentum δ functions

$$= \int\frac{d^3k\,d^3q}{(2\pi)^3 2\sqrt{\omega_{\vec k}\omega_{\vec q}}}\Big(-\frac{1}{2}\omega_{\vec k}\omega_{\vec q}\left[(a_{\vec k}a_{\vec q} + a^*_{\vec k}a^*_{\vec q})\delta(\vec k+\vec q)(2\pi)^3 - (a^*_{\vec k}a_{\vec q} + a_{\vec k}a^*_{\vec q})\delta(\vec k-\vec q)(2\pi)^3\right]$$

$$- \frac{1}{2}\vec k\cdot\vec q\left[(a_{\vec k}a_{\vec q} + a^*_{\vec k}a^*_{\vec q})\delta(\vec k+\vec q)(2\pi)^3 - (a^*_{\vec k}a_{\vec q} + a_{\vec k}a^*_{\vec q})\delta(\vec k-\vec q)(2\pi)^3\right]$$

$$+ \frac{\lambda v^2}{2}\left[(a_{\vec k}a_{\vec q} + a^*_{\vec k}a^*_{\vec q})\delta(\vec k+\vec q)(2\pi)^3 + (a^*_{\vec k}a_{\vec q} + a_{\vec k}a^*_{\vec q})\delta(\vec k-\vec q)(2\pi)^3\right]\Big)$$

$$= \int d^3k\Big[(a_{\vec k}a_{-\vec k} + a^*_{\vec k}a^*_{-\vec k})\frac{-\omega_{\vec k}\omega_{-\vec k} + \vec k^2 + \lambda v^2}{4\sqrt{\omega_{\vec k}\omega_{-\vec k}}} + (a^*_{\vec k}a_{\vec k} + a_{\vec k}a^*_{\vec k})\frac{\omega_{\vec k}^2 + \vec k^2 + \lambda v^2}{4\omega_{\vec k}}\Big]$$

for $\omega_{\vec k} = \sqrt{|\vec k|^2 + \lambda v^2}$, H_0^ψ has the desired form:

$$\int d^3x\,\mathcal{H}_0^\psi = \int d^3k\,(a^*_{\vec k}a_{\vec k} + a_{\vec k}a^*_{\vec k})\,\frac{1}{2}\omega_{\vec k}$$

. For $\mathcal{H}_0^\chi$, the calculation is as above with the substitutions: $a_{\vec p} \to b_{\vec p}\ \forall\vec p$, $\lambda v^2 \mapsto 0$, $\omega_{\vec p} \mapsto \varepsilon_{\vec p}\ \forall\vec p$

We'd get $\varepsilon_{\vec k} = \sqrt{|\vec k|^2}$. Given these dispersion relations it follows that:

$$\boxed{\begin{aligned}m_\psi &= \sqrt{\lambda}\,v\\ m_\chi &= 0\end{aligned}}$$

c) $A = \langle 0|\,b_{\vec q'}b_{\vec p'}\,H_{int}\,b^*_{\vec p}b^*_{\vec q}\,|0\rangle = ?$

From (a) we know that

$$H_{int} = \int\Big\{\frac{\lambda}{16}(\psi^4 + \chi^4 + 2\chi^2\psi^2) + \frac{\lambda v\sqrt{2}}{4}(\psi^3 + \chi^2\psi)\Big\}d^3x$$

To 1st order in perturbation theory, the relevant contribution to A comes from the $\int d^3x\,\frac{\lambda}{16}\chi^4(x)$ interaction term (the other interaction terms have the form $c_n\int d^3x\,\psi^n(x)$ and $d_m\int d^3x\,\psi^m(x)\chi^2(x)$. The 1st of these involves no χ fields so the amplitude of interest will be proportional to $\delta(\vec p-\vec p')\delta(\vec q-\vec q')$. For odd m,

$$\langle 0|b_{\vec q'}b_{\vec p'}\int d^3x\; d_m\psi^m(x)\chi^2(x)\,b^*_{\vec p}b^*_{\vec q}|0\rangle = 0$$

since the # of creation / destruction op. for ψ can't be the same. For even m, the non-zero contributions to the above amplitude will be proportional to

$$\int d^3k\,\langle 0|b_{\vec q'}b_{\vec p'}\,b^*_{\vec k}b_{\vec k}\,b^*_{\vec p}b^*_{\vec q}|0\rangle$$

$$= \int d^3k\;\delta(\vec k-\vec p)\,\delta(\vec p'-\vec k)\,\delta(\vec q-\vec q')$$

$$= \delta(\vec p-\vec p')\,\delta(\vec q-\vec q')$$ — this doesn't describe scattering since no momentum change happens.)

So to 1st order the desired amplitude is:

$$* \quad \langle 0|b_{\vec q'}b_{\vec p'}\int d^3x\,\frac{\lambda}{16}\chi^4(x)\,b^*_{\vec p}b^*_{\vec q}|0\rangle$$

$$= \langle 0|b_{\vec q'}b_{\vec p'}\int d^3x\,\frac{\lambda}{16}\int\frac{d^3k_1\,d^3k_2\,d^3k_3\,d^3k_4}{(2\pi)^6\,2^2\sqrt{\varepsilon_{\vec k_1}\varepsilon_{\vec k_2}\varepsilon_{\vec k_3}\varepsilon_{\vec k_4}}}$$

$$\left(b_{\vec k_1}e^{-i\varepsilon_{\vec k_1}t + i\vec k_1\cdot\vec x} + b^*_{\vec k_1}e^{i\varepsilon_{\vec k_1}t - i\vec k_1\cdot\vec x}\right)$$

$$\left(b_{\vec k_2}e^{-i\varepsilon_{\vec k_2}t + i\vec k_2\cdot\vec x} + b^*_{\vec k_2}e^{i\varepsilon_{\vec k_2}t - i\vec k_2\cdot\vec x}\right)$$

$$\left(b_{\vec k_3}e^{-i\varepsilon_{\vec k_3}t + i\vec k_3\cdot\vec x} + b^*_{\vec k_3}e^{i\varepsilon_{\vec k_3}t - i\vec k_3\cdot\vec x}\right)$$

$$\left(b_{\vec k_4}e^{-i\varepsilon_{\vec k_4}t + i\vec k_4\cdot\vec x} + b^*_{\vec k_4}e^{i\varepsilon_{\vec k_4}t - i\vec k_4\cdot\vec x}\right)b^*_{\vec p}b^*_{\vec q}|0\rangle$$

The non-zero contributions to the above can be written as (where (a,b,c,d) are some permutation of (1,2,3,4))

$$\langle 0|b_{\vec q'}b_{\vec p'}\int d^3x\,\frac{\lambda}{16}\int\frac{d^3k_a\,d^3k_b\,d^3k_c\,d^3k_d}{(2\pi)^6\,2^2\sqrt{\varepsilon_{\vec k_a}\varepsilon_{\vec k_b}\varepsilon_{\vec k_c}\varepsilon_{\vec k_d}}}\,b^*_{\vec k_a}b^*_{\vec k_b}\,b_{\vec k_c}b_{\vec k_d}\;b^*_{\vec p}b^*_{\vec q}|0\rangle$$
$$\cdot\, e^{i(\varepsilon_{\vec k_a} + \varepsilon_{\vec k_b} - \varepsilon_{\vec k_c} - \varepsilon_{\vec k_d})t}\,e^{-i(\vec k_a + \vec k_b - \vec k_c - \vec k_d)\vec x}$$

integrating over x gives a momentum δ function

$$= \left(\frac{\lambda}{16}\int\frac{d^3k_a\,d^3k_b\,d^3k_c}{(2\pi)^3\,4\sqrt{\varepsilon_{\vec k_a}\varepsilon_{\vec k_b}\varepsilon_{\vec k_c}\varepsilon_{\vec k_d}}}\cdot e^{i(\varepsilon_{\vec k_a} + \varepsilon_{\vec k_b} - \varepsilon_{\vec k_c} - \varepsilon_{\vec k_d})t}\,\langle 0|b_{\vec q'}b_{\vec p'}b^*_{\vec k_a}b^*_{\vec k_b}\,b_{\vec k_c}b_{\vec k_d}b^*_{\vec p}b^*_{\vec q}|0\rangle\right)\Big|_{\vec k_d = \vec k_a + \vec k_b - \vec k_c}$$

$$= \Big(\frac{\lambda}{16}\int\frac{d^3k_a\,d^3k_b\,d^3k_c}{(2\pi)^3\,4\sqrt{\varepsilon_{\vec k_a}\varepsilon_{\vec k_b}\varepsilon_{\vec k_c}\varepsilon_{\vec k_d}}}\cdot e^{i(\varepsilon_{\vec k_a} + \varepsilon_{\vec k_b} - \varepsilon_{\vec k_c} - \varepsilon_{\vec k_d})t}$$
$$\big\{\delta(\vec p-\vec k_d)\delta(\vec q-\vec k_c)\delta(\vec q'-\vec k_b)\delta(\vec p'-\vec k_a)$$
$$+ \delta(\vec q-\vec k_d)\delta(\vec p-\vec k_c)\delta(\vec q-\vec k_b)\delta(\vec p-\vec k_a)$$
$$+ \delta(\vec p-\vec k_d)\delta(\vec q-\vec k_c)\delta(\vec q'-\vec k_a)\delta(\vec p'-\vec k_b)$$
$$+ \delta(\vec q-\vec k_d)\delta(\vec p-\vec k_c)\delta(\vec q-\vec k_a)\delta(\vec p-\vec k_b)\big\}\Big)\Big|_{\vec k_d = \vec k_a + \vec k_b - \vec k_c}$$

$$= \frac{\lambda}{16}\,\frac{\delta(\vec q + \vec p - \vec p' - \vec q')}{(2\pi)^3\cdot\cancel{4}\sqrt{|\vec p||\vec p'||\vec q||\vec q'|}}\cdot\cancel{4}\,e^{i(|\vec p'| + |\vec q'| - |\vec p| - |\vec q|)t}$$

Going back to * we see that there are $\binom{4}{2} = 6$ distinct ways to chose the values of a,b,c,d.** To 1st order in pert. theory, the amplitude is then

$$A'(\vec p,\vec q \to \vec p',\vec q') = \frac{3\lambda}{2}\,\frac{(2\pi)^3\,\delta(\vec q + \vec p - \vec p' - \vec q')\,e^{i(|\vec p'| + |\vec q'| - |\vec p| - |\vec q|)t}}{((2\pi)^{3/2}\,2)^{4/2}\sqrt{|\vec p||\vec p'||\vec q||\vec q'|}}$$

If we also integrate over time, we get

$$A(\vec p,\vec q \to \vec p',\vec q') = \frac{3\lambda}{2}\,\frac{(2\pi)^4\,\delta^4(q' + p' - q - p)}{((2\pi)^{3/2}\,2)^2\sqrt{|\vec p||\vec p'||\vec q||\vec q'|}}$$

** For some a,b,c,d the relevant operators come to us as $b_{k_c}b_{k_d}b^*_{k_a}b^*_{k_b}$, again the additional terms we'd get from this (that come from the commutators of the bs) end up not mediating the process of interest and can be neglected.

At 2nd order in perturbation theory, multiple terms in $T(H_{int}(x)\,H_{int}(y))$ contribute to the desired amplitude ($T(\chi^4(x)\,\chi^4(y))$, $T(\chi^2(x)\psi^2(x)\,\chi^2(y)\psi^2(y))$, $T(\chi^2(x)\psi(x)\,\chi^2(y)\psi(y))$) however this last term will be dominant since it gives a contribution of order λ in the low energy limit (when $|\vec p|, |\vec q|, |\vec p'|, |\vec q'| \ll m_\psi$).

To see this, notice that the contribution from $T(\psi(x)\psi(y))$ will have the form

$$\int\frac{d^3l\,d^3l'}{(2\pi)^3 2\sqrt{\omega_l\omega_{l'}}}\;\delta(\vec l - \vec l')\;T\left(e^{i(lx - l'y)}\right)$$

$$= \int\frac{d^3l}{(2\pi)^3 2\sqrt{\omega_l}}\left(\theta(x^0 - y^0)\,e^{il(x-y)} + \theta(y^0 - x^0)\,e^{-il(x-y)}\right)$$

$$= \int\frac{d^4l}{(2\pi)^4}\,\frac{-i}{l^2 + \lambda v^2}\,e^{il(x-y)}$$

Here we expand $T(\psi(x)\,\psi(y))$ in terms of the momenta $\vec l, \vec l'$. Since $\psi(x)$ can only be contracted with $\psi(y)$, the desired matrix element will get a factor of $\delta(\vec l - \vec l')$.

The integrals over x,y will impose momentum conservation for the fields at x,y respectively. This will give a δ function that is non-zero only when l is a sum or difference of two of the initial / final momenta.

When the initial / final momenta can be neglected $l \approx 0$ and the λ dependence of the matrix element will be

$$\propto \int d^4l\,\frac{(\lambda v)^2}{l^2 + \lambda v^2}\,\delta^4(l)$$

from Hint

$$= \lambda$$

not λ^2, as we'd expect!