1. a) We have:

$$\Big(-\frac{\nabla^2}{2m} + \underbrace{V(\vec{x})}_{=V_0}\Big) u_R(\vec{x}) = \omega_R u_R(\vec{x})$$

and

$$u_R(\vec{x}=0) = u_R(\vec{x}=L\vec{e}_x) = u_R(\vec{x}=L\vec{e}_y)$$
$$= u_R(\vec{x}=L\vec{e}_z) = 0$$

This is a separable differential equation, we use the ansatz:

$$u_R(\vec{x}) = X(x)\,Y(y)\,Z(z)$$

$$\Big(-\frac{1}{2m}\Big(\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z}\Big) + (V_0-\omega_R)\Big) XYZ = 0$$

since we want $u \neq 0$

$$+\underbrace{\frac{1}{2m}\frac{Z''}{Z}}_{\text{function of } z} = \overbrace{-\frac{1}{2m}\Big(\frac{X''}{X}+\frac{Y''}{Y}\Big) + (V_0-\omega_R)}^{\text{function of } x,y \text{ only}}$$

$$Z''(z) - 2m\lambda_1 Z(z) = 0$$

$$Z(z) = \begin{cases} A_1 e^{\sqrt{2m\lambda_1}\,z} + A_2 e^{-\sqrt{2m\lambda_1}\,z} & ,\ \lambda_1 > 0 \\ A_1 e^{i\sqrt{2m|\lambda_1|}\,z} + A_2 e^{-i\sqrt{2m|\lambda_1|}\,z} & ,\ \lambda_1 < 0 \\ A_1 + A_2 z & ,\ \lambda_1 = 0 \end{cases}$$

- Boundary conditions: $Z(0) = Z(L) = 0$:

$$A_1 + A_2 = 0$$
$$A_1 e^{\sqrt{2m\lambda_1}L} + A_2 e^{-\sqrt{2m\lambda_1}L} = 0 \quad ,\ \lambda_1 > 0$$

$$\left.\begin{array}{l} A_1 = -A_2 \\ 2A_2 \sinh(-\sqrt{2m\lambda_1}L) = 0 \end{array}\right\} \Rightarrow A_1 = A_2 = 0 \quad \times$$

$$A_1 + A_2 = 0$$
$$A_1 e^{i\sqrt{2m|\lambda_1|}L} + A_2 e^{-i\sqrt{2m|\lambda_1|}L} = 0 \quad ,\ \lambda_1 < 0$$

$$\left.\begin{array}{l} A_1 = -A_2 \\ A_2\, 2i \sin(-\sqrt{2m|\lambda_1|}L) = 0 \end{array}\right\} \Rightarrow \lambda_1 = -\frac{1}{2m}\Big(\frac{n\pi}{L}\Big)^2, \quad n \in \mathbb{Z}$$

$$\left.\begin{array}{l} A_1 = 0 \\ A_2 L = 0 \quad ,\ \lambda_1 = 0 \end{array}\right\} \Rightarrow A_1 = A_2 = 0$$

- The only non-trivial solution is then:

$$Z_{r_z}(z) = A_z \sin\Big(\frac{r_z \pi z}{L}\Big) \quad ,\ r_z > 0,\ r_z \in \mathbb{N} \qquad (r_z \neq 0 \text{, since } Z_0(z) = 0)$$

Next,

$$-\frac{1}{2m}\Big(\frac{X''}{X} + \frac{Y''}{Y}\Big) + (V_0 - \omega_R) = -\frac{1}{2m}\Big(\frac{r_z\pi}{L}\Big)^2$$

$$\underbrace{\frac{1}{2m}\frac{Y''}{Y}}_{\text{function of } y \text{ only}} = \underbrace{-\frac{1}{2m}\frac{X''}{X} + V_0 - \omega_R + \frac{1}{2m}\Big(\frac{r_z\pi}{L}\Big)^2}_{\text{function of } x \text{ only}}$$

$$Y''(y) - 2m\lambda_2 Y(y) = 0 \qquad \text{(same equation as above, same boundary conditions)}$$

$$Y_r(y) = A_y \sin\Big(\frac{r_y\pi y}{L}\Big) \quad ,\ \lambda_2 = -\frac{1}{2m}\Big(\frac{r_y\pi}{L}\Big)^2, \quad r_y \in \mathbb{N},\ r_y > 0$$

Finally,

$$-\frac{1}{2m}\frac{X''}{X} + V_0 - \omega_R + \frac{1}{2m}\Big(\frac{\pi}{L}\Big)^2 (r_z^2 + r_y^2) = 0$$

$$X''(x) - \Big(2m(V_0-\omega_R) + \Big(\frac{\pi}{L}\Big)^2(r_y^2+r_z^2)\Big) X(x) = 0$$

(same type of eqn. as above, same bound. conditions)

$$X_e(x) = A_x \sin\Big(\frac{r_x \pi x}{L}\Big) \quad ,\ r_x \in \mathbb{N},\ r_x > 0$$

$$\lambda_3 \equiv V_0 - \omega_R + \Big(\frac{\pi}{L}\Big)^2 (r_z^2 + r_y^2)\frac{1}{2m} = -\frac{1}{2m}\Big(\frac{r_x\pi}{L}\Big)^2$$

$$\boxed{\omega_R = V_0 + \frac{1}{2m}\Big(\frac{\pi}{L}\Big)^2 (r_x^2 + r_y^2 + r_z^2)}$$

- The solution depends on the quantum numbers $r_x, r_y, r_z \in \mathbb{N}$ and is given by

normalization constant

$$\boxed{u_{r_x,r_y,r_z}(\vec{x}) = \sqrt{\frac{8}{L^3}} \sin\Big(\frac{r_x\pi x}{L}\Big) \sin\Big(\frac{r_y\pi y}{L}\Big) \sin\Big(\frac{r_z\pi z}{L}\Big)}$$

(b) If the particles are bosons the 4 lowest energy eigenstates / eigenvalues are:

- $|0\rangle$, the no particle state with energy E_0
- $|N(1,1,1)=1\rangle$, a 1 particle state with $R_x = R_y = R_z = 1$ and energy $E_0 + V_0 + \frac{3}{2m}\big(\frac{\pi}{L}\big)^2$
- $|N(1,1,2)=1\rangle$ (or $|N(1,2,1)=1\rangle$, $|N(2,1,1)=1\rangle$) a 1 particle state with $|\vec{R}| = \sqrt{1^2+1^2+2^2} = \sqrt{6}$ and energy $E_0 + V_0 + \frac{6}{2m}\big(\frac{\pi}{L}\big)^2$
- For the 4th lowest energy state, we have $|N(1,1,1)=2\rangle$ a 2 particle state, with $|\vec{R}| = \sqrt{3}$ each and energy

$$E_0 + 2V_0 + \frac{6}{2m}\Big(\frac{\pi}{L}\Big)^2$$

or $|N(1,2,2)=1\rangle$ (or $|N(2,1,2)=1\rangle$, $|N(2,2,1)=1\rangle$) a 1 particle state with $|\vec{R}| = 3$ and energy $E_0 + V_0 + \frac{9}{2m}\big(\frac{\pi}{L}\big)^2$

if $V_0 < \frac{3}{2m}\big(\frac{\pi}{L}\big)^2$ then the state $|N(1,1,1)=2\rangle$ has lower energy otherwise they either have the same energy ($V_0 = \frac{3}{2m}\big(\frac{\pi}{L}\big)^2$) or the 2nd state has lower energy ($V_0 > \frac{3}{2m}\big(\frac{\pi}{L}\big)^2$).

- If the particles are fermions, the first 3 lowest energy states are the same as before.
The 4th lowest energy state is the 1 particle state with $|\vec{R}|=3$ and energy $E_0 + V_0 + \frac{9}{2m}\big(\frac{\pi}{L}\big)^2$ $|N(1,2,2)=1\rangle$ since two fermions can't be in the same state (as in $|N(1,1,1)=2\rangle$).

(c)

$$A(\vec{x}) = \sum_R a_R u_R(\vec{x})$$

$$\langle\{N_R\}|A(\vec{x})|\{N_R\}\rangle$$

$$= \sum_{R'} \langle N_{R_n},\dots N_{R_1}| a_{R'} u_{R'}(\vec{x}) | N_{R_1},\dots N_{R_n}\rangle$$

$$= \sum_{R'}\sum_{j=1}^{n} u_{R'}(\vec{x})\, \delta_{R',R_j} \sqrt{N_{R_j}}\, \underbrace{\langle N_{R_n},\dots,N_{R_j},\dots,N_{R_1} | N_{R_1},\dots,N_{R_j}-1,\dots N_{R_n}\rangle}_{=0}$$

$$\boxed{=0}$$

similarly, $\langle\{N_R\}|A^*(\vec{x})|\{N_R\}\rangle = 0$

$$\langle\{N_R\}|A^*(\vec{x})A(\vec{y})|\{N_R\}\rangle$$

$$= \sum_{R'}\sum_{R''} u^*_{R'}(\vec{x})\, u_{R''}(\vec{y}) \langle N_{R_n},\dots N_{R_1}| a^*_{R'} a_{R''} | N_{R_1},\dots N_{R_n}\rangle$$

$$= \sum_{R',R''} u^*_{R'}(\vec{x})\, u_{R''}(\vec{y}) \sum_{j=1}^{n}\sum_{j'=1}^{n} \delta_{R'',R_j}\delta_{R',R_{j'}} \sqrt{N_{R_j}}\sqrt{N_{R_{j'}}+1}$$
$$\cdot \underbrace{\langle N_{R_n},\dots N_{R_1}| N_{R_1},\dots N_{R_j}-1, N_{R_{j'}}+1,\dots N_{R_n}\rangle}_{=\delta_{R_j,R_{j'}}}$$

(the overlap vanishes if $R_j \neq R_{j'}$, and is 1 if $R_j = R_{j'}$ since the state is normalized)

$$\boxed{= \sum_{j=1}^{n} u^*_{R_j}(\vec{x})\, u_{R_j}(\vec{y})\, N_{R_j}}$$

The correlation function is then

$$\boxed{\begin{array}{l}\langle A^*(\vec{x})A(\vec{y})\rangle - \langle A^*(\vec{x})\rangle\langle A(\vec{y})\rangle \\ = \sum_{j=1}^{n} u^*_{R_j}(\vec{x})\, u_{R_j}(\vec{y})\, N_{R_j}\end{array}}$$

(d) There are no particles in the ground state, so the above evaluates to 0 as does the variance which is equal to the correlation function when $\vec{x} = \vec{y}$.

(e)

$$\langle\{\alpha_R\}|A(\vec{x})|\{\alpha_R\}\rangle$$

$$= \sum_{R'} u_{R'}(\vec{x}) \langle\{\alpha_R\}| \overbrace{a_{R'}|\{\alpha_R\}\rangle}^{=\alpha_{R'}|\{\alpha_R\}\rangle}$$

$$= \sum_{R'} u_{R'}(\vec{x})\,\alpha_{R'} \underbrace{\langle\{\alpha_R\}|\{\alpha_R\}\rangle}_{=1}$$

$$\boxed{= \sum_{R'} u_{R'}(\vec{x})\,\alpha_{R'}}$$

$$\langle\{\alpha_R\}|A^*(\vec{x})A(\vec{x})|\{\alpha_R\}\rangle$$

$$= \sum_{R',R''} u^*_{R'}(\vec{x})\, u_{R''}(\vec{x}) \underbrace{\langle\{\alpha_R\}|a^*_{R'}}_{\alpha^*_{R'}\langle\{\alpha_R\}|} \overbrace{a_{R''}|\{\alpha_R\}\rangle}^{\alpha_{R''}|\{\alpha_R\}\rangle}$$

$$\boxed{= \sum_{R',R''} u^*_{R'}(\vec{x})\, u_{R''}(\vec{x})\, \alpha^*_{R'}\alpha_{R''}}$$

- The variance is:

$$\langle\{\alpha_R\}|A^*(\vec{x})A(\vec{x})|\{\alpha_R\}\rangle - |\langle\{\alpha_R\}|A(\vec{x})|\{\alpha_R\}\rangle|^2$$

$$= \sum_{R',R''} u^*_{R'}(\vec{x})\, u_{R''}(\vec{x})\, \alpha^*_{R'}\alpha_{R''}$$
$$-\Big(\sum_{R'} u^*_{R'}(\vec{x})\,\alpha^*_{R'}\Big)\sum_{R''} u_{R''}(\vec{x})\,\alpha_{R''}$$

$$\boxed{=0}$$

$$\langle\{\alpha_R\}|H|\{\alpha_R\}\rangle$$

$$= \langle\{\alpha_R\}| E_0 + \sum_{R'} \omega_{R'} a^*_{R'} \overbrace{a_{R'}|\{\alpha_R\}\rangle}^{\alpha_{R'}|\{\alpha_R\}\rangle}$$

$$\boxed{= E_0 + \sum_{R'} \omega_{R'} |\alpha_{R'}|^2}$$

$$\langle\{\alpha_R\}|H^*H|\{\alpha_R\}\rangle$$

$$= \langle\{\alpha_R\}| E_0^2 + 2E_0\sum_{R'}\omega_{R'} a^*_{R'}a_{R'}$$
$$+ \sum_{R',R''} \omega_{R'}\omega_{R''} a^*_{R'}a_{R'}a^*_{R''}a_{R''} |\{\alpha_R\}\rangle$$

$$= E_0^2 + 2E_0\sum_{R'}\omega_{R'}|\alpha_{R'}|^2$$
$$+ \sum_{R',R''}\omega_{R'}\omega_{R''} \underbrace{\langle\{\alpha_R\}|a^*_{R'}}_{\langle\{\alpha_R\}|\alpha^*_{R'}} \overbrace{a_{R'}a^*_{R''}}^{\delta_{R',R''}+a^*_{R''}a_{R'}} \overbrace{a_{R''}|\{\alpha_R\}\rangle}^{\alpha_{R''}|\{\alpha_R\}\rangle}$$

$$= E_0^2 + 2E_0\sum_{R'}\omega_{R'}|\alpha_{R'}|^2 + \sum_{R',R''}\omega_{R'}\omega_{R''}\alpha_{R''}\alpha^*_{R'}\Big(\delta_{R',R''}$$
$$+ \underbrace{\langle\{\alpha_R\}|a^*_{R''}a_{R'}|\{\alpha_R\}\rangle}_{=\alpha_{R'}\alpha^*_{R''}}\Big)$$

$$\boxed{\begin{array}{l} = E_0^2 + \sum_{R'}\big(2E_0\omega_{R'}|\alpha_{R'}|^2 + \omega_{R'}^2|\alpha_{R'}|^2\big) \\ \quad + \sum_{R',R''}\omega_{R'}\omega_{R''}|\alpha_{R'}|^2|\alpha_{R''}|^2 \end{array}}$$

the variance is:

$$\langle H^*H\rangle - |\langle H\rangle|^2$$

$$= E_0^2 + \sum_{R'}\big(2E_0\omega_{R'}|\alpha_{R'}|^2 + \omega_{R'}^2|\alpha_{R'}|^2\big)$$
$$+ \sum_{R',R''}\omega_{R'}\omega_{R''}|\alpha_{R'}|^2|\alpha_{R''}|^2 - \Big(E_0 + \sum_{R'}\omega_{R'}|\alpha_{R'}|^2\Big)^2$$

$$= \cancel{E_0^2} + \sum_{R'}\big(\cancel{2E_0\omega_{R'}|\alpha_{R'}|^2} + \omega_{R'}^2|\alpha_{R'}|^2\big)$$
$$+ \cancel{\sum_{R',R''}\omega_{R'}\omega_{R''}|\alpha_{R'}|^2|\alpha_{R''}|^2}$$
$$-\Big(\cancel{E_0^2} + \cancel{2E_0\sum_{R'}\omega_{R'}|\alpha_{R'}|^2} + \cancel{\sum_{R',R''}\omega_{R'}\omega_{R''}|\alpha_{R'}|^2|\alpha_{R''}|^2}\Big)$$

$$\boxed{= \sum_{R'}\omega_{R'}^2|\alpha_{R'}|^2}$$

(f)

$$\tilde{H} = E_0 + J(a^*_{R_0} + a_{R_0}) + \sum_R \omega_R a^*_R a_R$$
$$= E_0 + J(a^*_{R_0} + a_{R_0}) + \omega_{R_0}a^*_{R_0}a_{R_0} + \sum_{R\neq R_0}\omega_R a^*_R a_R$$

To simplify this hamiltonian, we introduce a new operator

$$b_{R_0} \equiv a_{R_0} - \alpha \quad ,\ a_{R_0} = b_{R_0} + \alpha$$

where α is a constant. Plugging this into $\tilde{H}$, we get

$$\tilde{H} = E_0 + J(b^*_{R_0} + b_{R_0} + \alpha + \alpha^*)$$
$$+ \omega_{R_0}(b^*_{R_0} + \alpha^*)(b_{R_0} + \alpha) + \sum_{R\neq R_0}\omega_R a^*_R a_R$$
$$= E_0 + b_{R_0}(J + \alpha^*\omega_{R_0}) + b^*_{R_0}(J + \alpha\omega_{R_0})$$
$$+ J(\alpha+\alpha^*) + \omega_{R_0}|\alpha|^2 + \omega_{R_0}b^*_{R_0}b_{R_0} + \sum_{R\neq R_0}\omega_R a^*_R a_R$$
$$= E_0 - \frac{J^2}{\omega_{R_0}} + \omega_{R_0}b^*_{R_0}b_{R_0} + \sum_{R\neq R_0}\omega_R a^*_R a_R$$

where in the last line we simplify $\tilde{H}$ by choosing

$$\alpha = \alpha^* = -\frac{J}{\omega_{R_0}}$$

It's now clear that the eigenstates of $\tilde{H}$ have the form*

$$(b^*_{R_0})^n |\{N_{R\neq R_0}\}\rangle$$

where n is any non-negative integer.

(* note that $[b_{R_0}, b^*_{R_0}] = [a_{R_0}, a^*_{R_0}] = 1$ so $(b^*_{R_0})^n|0\rangle$ is an eigenstate of $b^*_{R_0}b_{R_0}$ if $b_{R_0}|0\rangle = 0$)

The state $(b^*_{R_0})^n|\{N_{R\neq R_0}\}\rangle$ has energy

eigenstate of $\tilde{H}$

$$\boxed{\begin{array}{l}\tilde{H}(b^*_{R_0})^n|\{N_{R\neq R_0}\}\rangle \\ = \overbrace{\Big(E_0 - \frac{J^2}{\omega_{R_0}} + \omega_{R_0}n + \sum_{R\neq R_0}N_R\omega_R\Big)}^{\text{eigenvalue of } \tilde{H}}(b^*_{R_0})^n|\{N_{R\neq R_0}\}\rangle\end{array}}$$

Note that the vacuum is a coherent state,

$$b_{R_0}|0\rangle = \Big(a_{R_0} + \frac{J}{\omega_{R_0}}\Big)|0\rangle = 0$$

$$a_{R_0}|0\rangle = -\frac{J}{\omega_{R_0}}|0\rangle.$$