1. (a)

$$[a_p, a_q^*] = \delta_{pq}$$

If $b_p \equiv a_p^*$, then $b_p^* = a_p$ and it follows:

$$[b_p, b_q^*] = [a_p^*, a_q] = -[a_q, a_p^*] = -\delta_{pq}$$

· for $p=q$:

$$\langle 0 | [b_p, b_p^*] | 0 \rangle = \overbrace{-\langle 0|0\rangle}^{\text{this is } \leq 0}$$
$$= \langle 0 | b_p b_p^* - \overbrace{b_p^* b_p | 0 \rangle}^{=0}$$

$\langle 0 | b_p b_p^* | 0 \rangle$ is equal to the norm of the state $b_p^* | 0 \rangle$, which has to be ≥ 0!

(b) $\{c_p, c_q^*\} = \delta_{pq}$

$\tilde{c}_p \equiv c_p^*$

$\hookrightarrow \{\tilde{c}_p, \tilde{c}_q^*\} = \{c_p^*, c_q\} = \delta_{pq}$

if $\tilde{c}_p |0\rangle = 0$

$\langle 0 | \{\tilde{c}_p, \tilde{c}_p^*\} | 0 \rangle = \langle 0 | 0 \rangle = \langle 0 | \tilde{c}_p \tilde{c}_p^* | 0 \rangle$ no inconsistencies here

2. $$H = \sum_p \left[\varepsilon(p) a_p^* a_p + \frac{1}{2} \gamma(p) (a_p a_p + a_p^* a_p^*) \right]$$

(a)

$a_p \equiv b_p \cosh \beta_p + b_p^* \sinh \beta_p$

$a_p^* \equiv b_p^* \cosh \beta_p + b_p \sinh \beta_p$

$\hookrightarrow \cosh \beta_p a_p - \sinh \beta_p a_p^*$

$$= b_p (\overbrace{\cosh^2 \beta_p - \sinh^2 \beta_p}^{=1}) + b_p^* \cosh \beta_p \sinh \beta_p \overbrace{(1-1)}^{=0}$$

$$\hookrightarrow \boxed{\begin{aligned} b_p &= \cosh \beta_p a_p - \sinh \beta_p a_p^* \\ b_p^* &= \cosh \beta_p a_p^* - \sinh \beta_p a_p \end{aligned}}$$

$[b_p, b_q]$

$= [\cosh \beta_p a_p - \sinh \beta_p a_p^*, \cosh \beta_q a_q - \sinh \beta_q a_q^*]$

$= -\sinh \beta_p \cosh \beta_q \underbrace{[a_p^*, a_q]}_{=-\delta_{p,q}} - \sinh \beta_q \cosh \beta_p \underbrace{[a_p, a_q^*]}_{=\delta_{p,q}}$

$= 0$

$[b_p, b_q^*]$

$= [\cosh \beta_p a_p - \sinh \beta_p a_p^*, \cosh \beta_q a_q^* - \sinh \beta_q a_q]$

$= \cosh \beta_p \cosh \beta_q \overbrace{[a_p, a_q^*]}^{=\delta_{p,q}} + \sinh \beta_p \sinh \beta_q \underbrace{[a_p^*, a_q]}_{=-\delta_{p,q}}$

$= (\cosh \beta_p^2 - \sinh^2 \beta_p) \delta_{p,q}$

$\hookrightarrow \cosh \beta_p^2 - \sinh^2 \beta_p = 1$ (automatically satisfied)

(b)

$a_p^* a_p = (b_p^* \cosh \beta_p + b_p \sinh \beta_p)(b_p \cosh \beta_p + b_p^* \sinh \beta_p)$

$= b_p^* b_p \cosh^2 \beta_p + \sinh \beta_p \cosh \beta_p (b_p b_p + b_p^* b_p^*)$

$+ \underbrace{b_p b_p^*}_{=1 + b_p^* b_p} \sinh^2 \beta_p$

$= b_p^* b_p (\cosh^2 \beta_p + \sinh^2 \beta_p) + \sinh^2 \beta_p$

$+ (b_p b_p + b_p^* b_p^*) \sinh \beta_p \cosh \beta_p$

$a_p a_p = (b_p \cosh \beta_p + b_p^* \sinh \beta_p)(b_p \cosh \beta_p + b_p^* \sinh \beta_p)$

$= b_p b_p \cosh^2 \beta_p + b_p^* b_p^* \sinh^2 \beta_p$

$+ \cosh \beta_p \sinh \beta_p \underbrace{(b_p b_p^* + b_p^* b_p)}_{= 2 b_p^* b_p + 1}$

$a_p^* a_p^* = (a_p a_p)^* = b_p^* b_p^* \cosh^2 \beta_p + b_p b_p \sinh^2 \beta_p$

$+ \cosh \beta_p \sinh \beta_p (2 b_p^* b_p + 1)$

· plugging this into H, we get:

$$H = \sum_p \{ b_p^* b_p (\varepsilon(p) (\cosh^2 \beta_p + \sinh^2 \beta_p)$$
$$+ 2 \gamma(p) \cosh \beta_p \sinh \beta_p)$$
$$+ (b_p b_p + b_p^* b_p^*) (\varepsilon(p) \sinh \beta_p \cosh \beta_p$$
$$+ \frac{1}{2} \gamma(p) (\cosh^2 \beta_p + \sinh^2 \beta_p))$$
$$+ \varepsilon(p) \sinh^2 \beta_p + \gamma(p) \cosh \beta_p \sinh \beta_p \}$$

If $H = E_0 + \sum_p E(p) b_p^* b_p$, it follows that:

$$2 \varepsilon(p) \underbrace{\sinh \beta(p) \cosh \beta(p)}_{= \frac{1}{2} \sinh(2\beta(p))} + \gamma(p) \underbrace{(\cosh^2 \beta(p) + \sinh^2 \beta(p))}_{= \cosh(2\beta(p))} = 0$$

$$\hookrightarrow \tanh(2\beta(p)) = -\frac{\gamma(p)}{\varepsilon(p)}$$

$$\boxed{\begin{aligned} &\beta(p) = \frac{1}{2} \operatorname{arctanh}\left(-\frac{\gamma(p)}{\varepsilon(p)}\right) \\ &\text{and} \\ &E_0 = \sum_p \left(\varepsilon(p) \sinh^2 \beta(p) + \gamma(p) \cosh \beta(p) \sinh \beta(p)\right) \\ &E(p) = \varepsilon(p) \cosh(2\beta(p)) + \gamma(p) \sinh(2\beta(p)) \end{aligned}}$$

(c) $b_p | \Omega \rangle = 0$

$\langle \Omega | a_p^* a_p | \Omega \rangle$

$= \langle \Omega | b_p^* b_p (\cosh^2 \beta_p + \sinh^2 \beta_p) + \sinh^2 \beta_p$

$+ (b_p b_p + b_p^* b_p^*) \sinh \beta_p \cosh \beta_p | \Omega \rangle$

$= \sinh^2 \beta_p \langle \Omega | \Omega \rangle$

$\boxed{= \sinh^2 \beta_p}$

$\hookrightarrow$ if $|\Omega\rangle$ is normalized

3. $H = \frac{\omega}{2} (a^* a + a a^*)$

$a |\alpha\rangle = \alpha |\alpha\rangle$

$A = \frac{1}{2} (a + a^*)$

$\langle n | A | n \rangle = \frac{1}{2} \langle n | a + a^* | n \rangle$

$= \frac{1}{2} \langle n | (\sqrt{n} | n-1 \rangle + \sqrt{n+1} | n+1 \rangle)$

$\boxed{= 0}$

$\langle n | A^2 | n \rangle = \frac{1}{4} \langle n | a^2 + a a^* + a^* a + a^* a^* | n \rangle$

$= \frac{1}{4} \langle n | (\sqrt{n} \sqrt{n-1} | n-2 \rangle + (n+1) | n \rangle$

$+ n | n \rangle + \sqrt{n+1} \sqrt{n+2} | n+2 \rangle)$

$= \frac{2n+1}{4} \langle n | n \rangle$

$\boxed{= \frac{2n+1}{4}}$

the variance is $\langle n | A^2 | n \rangle - (\langle n | A | n \rangle)^2 = \frac{2n+1}{4}$