Physics 4Q03: Introduction to Quantum Field Theory

Midterm Exam

Thursday March 2, 2023.

This exam comprises four (4) questions and a formula sheet, for a total of three (3) pages. Answer all questions. Allowed aids: The attached formula sheet, a single-sided homespun formula sheet and a calculator. No other aids are allowed.

1. *(10 marks)*

   Fill in the missing factors of Boltzmann's constant, k_B, Planck's constant, $\hbar$, and the speed of light, c, in the following formulae:

   (a) $M_p^2 = 1/(8\pi G)$ where M_p is the reduced Planck mass and G is Newton's gravitational constant.

   M_p has dimensions of mass. Because eg $E = GMm/r$ is an energy G must have units energy-length/mass^2. So LHS has dimension (mass)^2 and RHS has dimension (mass)^2/(energy-length). $\hbar$ has dimension energy-time and c has dimension length/time, so $\hbar c$ has dimension energy-length. The formula with the right dimension therefore is

   $$M_p^2 = \frac{\hbar c}{8\pi G}$$

   (b) $T_U = a/(2\pi)$ where a is an acceleration and T_U is the Unruh temperature.

   T has dimensions of temperature so $k_B T$ has dimension energy (and so is also mass-(length/time)^2). a is an acceleration so has units length/time^2. So the mismatch between LHS and RHS is mass-length. $\hbar c$ has dimension energy-length (from previous question) and so $\hbar c/c^2 = \hbar/c$ has dimension mass-length. The formula with the right dimension therefore is

   $$k_B T = \frac{\hbar a}{2\pi c}$$

2. *(15 marks)* Consider a simple harmonic oscillator whose Hamiltonian is given by

   $$H = \omega\left(a^\star a + \frac{1}{2}\right), \tag{2.1}$$

   where, as usual $[a, a^\star] = 1$. A coherent state, $|\alpha\rangle$, for this system is given in terms of the occupation-number basis by the expression given in class:

   $$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\,|\,n\rangle\,. \tag{2.2}$$

   The position operator for this oscillator is given by

   $$x = \frac{a + a^\star}{\sqrt{2m\omega}} \tag{2.3}$$

(a) Compute the average and variance of the position in a coherent state

$$\overline{x} = \langle\alpha|x|\alpha\rangle\,, \qquad \text{and} \quad (\Delta x)^2 = \langle\alpha|(x-\overline{x})^2|\alpha\rangle\,. \tag{2.4}$$

Coherent states are defined by the condition $a|\alpha\rangle = \alpha|\alpha\rangle$ and this implies $\langle\alpha|a^\star = \langle\alpha|\alpha^*$.

Compute the mean:

$$\overline{x} = \frac{1}{\sqrt{2m\omega}}\langle\alpha|(a+a^\star)|\alpha\rangle = \frac{1}{\sqrt{2m\omega}}(\alpha+\alpha^*)\,.$$

where in the last equality a is taken to act on the state on the right and $a^\star$ acts on the state to the left.

Compute the variance: First use that

$$\langle(x-\bar{x})^2\rangle = \langle x^2\rangle - 2\bar{x}\langle x\rangle + \bar{x}^2 = \langle x^2\rangle - \bar{x}^2\,.$$

Next use that

$$\bar{x}^2 = \frac{(\alpha+\alpha^*)^2}{2m\omega}.$$

In $\langle\alpha|x^2|\alpha\rangle$ reorder any operators so that all the a^*'s are on the left and the a's on the right, so that we can use $\langle\alpha|a^\star = \langle\alpha|\alpha^*$ and $a|\alpha\rangle = \alpha|\alpha\rangle$. Re-ordering is done using $aa^\star = a^\star a + 1$. Therefore

$$\begin{aligned}\langle\alpha|x^2|\alpha\rangle &= \frac{\langle\alpha|(a+a^\star)^2|\alpha\rangle}{2m\omega} = \frac{\langle\alpha|(a^2+aa^\star+a^\star a+(a^\star)^2|\alpha\rangle}{2m\omega} \\ &= \frac{\langle\alpha|(a^2+1+2a^\star a+(a^\star)^2|\alpha\rangle}{2m\omega} = \frac{\alpha^2+1+2\alpha^*\alpha+(\alpha^*)^2}{2m\omega}\end{aligned}$$

and so subtracting $\bar{x}^2$ gives

$$\langle\alpha|x^2|\alpha\rangle - \bar{x}^2 = \frac{1}{2m\omega}\,.$$

3. *(15 marks)* Suppose a system of particles experiences a potential $V(x,y)$ that traps them to move only along a line (a cosmic string, say, or a condensed matter defect), which we choose to be the z axis. [The potential could be something like $V(x,y) = \frac{1}{2}m\omega^2(x^2+y^2)$, though you do not need to know its explicit form for this question.] What is important is that the single-particle states are described by their momentum p along the z axis, and two non-negative integers $r, s = 0, 1, \cdots$ that are quantum numbers related to the particle's confinement to the z axis.

Using discretely normalized momentum states, the Hamiltonian for the system is $H = H_{\text{free}} + H_{\text{int}}$ with

$$H_{\text{free}} = \sum_{rs}\sum_{p} \varepsilon_{rs}(p)\, a^\star_{rsp}a_{rsp} \tag{3.1}$$

and

$$H_{\text{int}} = \sum_{r_1,s_1,r_2,s_2,r_3,s_3} \sum_{p_1,p_2,p_3} \Big[U_{r_1 s_1 r_2 s_2 r_3 s_3}(p_1,p_2,p_3)\, a^\star_{r_1 s_1 p_1} a^\star_{r_2 s_2 p_2} a_{r_3 s_3 p_3} + \text{h.c.} \Big] \delta_{p_1+p_2-p_3} \,. \tag{3.2}$$

The operators a_{rsp} satisfy the commutation relation $[a_{rsp}, a^\star_{r's'p'}] = \delta_{rr'}\delta_{ss'}\delta_{pp'}$.

(a) Rewrite this Hamiltonian in terms of continuum-normalized states, for which

$$[\hat{a}_{rsp}, \hat{a}^\star_{r's'p'}] = \delta_{rr'}\delta_{ss'}\delta(p-p'),$$

where $\delta(p-p')$ is a Dirac delta function for the momentum in the z direction. When computing the density of states take the system's length in the z direction to be L and imagine L to be very large (as we did in class when doing these conversions). In particular show that the Hamiltonian becomes $H = H_{\text{free}} + H_{\text{int}}$ with

$$H_{\text{free}} = \sum_{rs} \int \mathrm{d}p\, \widehat{\varepsilon}_{rs}(p)\, \hat{a}^\star_{rsp}\hat{a}_{rsp} \tag{3.3}$$

and

$$H_{\text{int}} = \sum_{r_1,s_1,r_2,s_2,r_3,s_3} \int \mathrm{d}p_1\, \mathrm{d}p_2\, \mathrm{d}p_3 \Big[\widehat{U}_{r_1 s_1 r_2 s_2 r_3 s_3}(p_1,p_2,p_3)\, \hat{a}^\star_{r_1 s_1 p_1} \hat{a}^\star_{r_2 s_2 p_2} \hat{a}_{r_3 s_3 p_3} + \text{h.c.} \Big] \delta(p_1+p_2-p_3) \,. \tag{3.4}$$

How are the continuum coefficient functions $\widehat{\varepsilon}_{rs}(p)$ and $\widehat{U}_{r_1 s_1 r_2 s_2 r_3 s_3}(p_1,p_2,p_3)$ related to the functions $\varepsilon_{rs}(p)$ and $U_{r_1 s_1 r_2 s_2 r_3 s_3}(p_1,p_2,p_3)$ in the discrete formulation? In passing to the continuum from discrete states use that discrete states (with periodic boundary conditions) have levels $p = 2\pi n/L$ and so the number of states in an interval $\mathrm{d}p$ is $\mathrm{d}n = (L/2\pi)\mathrm{d}p$. Summations and delta functions therefore convert with

$$\sum_p = \frac{L}{2\pi}\int \mathrm{d}p \quad \text{and} \quad \delta_{pq} = \frac{2\pi}{L}\,\delta(p-q)$$

and so normalized states and destruction/creation operators are related by

$$|p\rangle = \sqrt{L/2\pi}\; |p) \qquad \text{and so} \qquad \hat{a}_p = \sqrt{L/2\pi}\; a_p.$$

Therefore

$$H_{\text{free}} = \sum_{rs}\sum_p \varepsilon_{rs}(p)\, a^\star_{rsp} a_{rsp} = \sum_{rs} \left(\frac{L}{2\pi} \int \mathrm{d}p \right) \varepsilon_{rs}(p) \left(\frac{2\pi}{L}\, \hat{a}^\star_{rsp}\hat{a}_{rsp} \right)$$

and so $\hat{\varepsilon}_{rs}(p) = \varepsilon_{rs}(p)$. Similarly

$$H_{\rm int} = \sum_{r_1\cdots s_3} \sum_{p_1,p_2,p_3} \Big[U_{r_1s_1r_2s_2r_3s_3}(p_1,p_2,p_3)\, a^\star_{r_1s_1p_1} a^\star_{r_2s_2p_2} a_{r_3s_3p_3} + \text{h.c.} \Big] \delta_{p_1+p_2-p_3}$$

$$= \sum_{r_1\cdots s_3} \left(\frac{L}{2\pi}\right)^3 \int {\rm d}p_1 {\rm d}p_2 {\rm d}p_3 \Big[U_{r_1s_1r_2s_2r_3s_3}(p_1,p_2,p_3)$$

$$\times \left(\frac{2\pi}{L}\right)^{3/2} \hat{a}^\star_{r_1s_1p_1} \hat{a}^\star_{r_2s_2p_2} \hat{a}_{r_3s_3p_3} + \text{h.c.} \Big] \frac{2\pi}{L}\, \delta(p_1+p_2-p_3)\,.$$

and so

$$\widehat{U}_{r_1s_1r_2s_2r_3s_3}(p_1,p_2,p_3) = \sqrt{\frac{L}{2\pi}}\; U_{r_1s_1r_2s_2r_3s_3}(p_1,p_2,p_3)\,.$$

Since U has dimensions energy it follows $\widehat{U}$ has dimensions of $(\text{energy})^{1/2}$.

(b) Suppose the single-particle energies are

$$\varepsilon_{rs}(p) = \frac{p^2}{2m} + \Big(r+s+1\Big)\omega$$

where m and ω are real and positive constants. Do you expect the particles described in this system to be stable? Briefly explain why or why not, and if you think they are not stable what do they decay into?

The basic interaction destroys a particle and creates two others with different values of their quantum numbers, and this allows any state (r,s,p) to decay into two states with smaller (r,s), provided energy and momentum conservation is possible. This means that the states will generically be unstable. Exceptions to this are the lowest-energy states $(0,0,p)$ who cannot lose any more oscillator energy.

(There is no need to compute a decay rate here, though there are bonus marks if you can estimate the decay lifetime for states you think can decay.)

From Fermi's golden rule the rate must be proportional to the interaction hamiltonian squared, so has a factor $|\widehat{U}|^2$. This has dimension energy and so is the same dimension as is the rate ${\rm d}\Gamma$, since this is inverse-time. Fermi's rule says

$$\Gamma \sim \int {\rm d}p\, {\rm d}q\, |\widehat{U}|^2\, \delta(p+q)\, \delta\left(\omega - \frac{q^2}{2m} - \frac{p^2}{2m}\right)$$

$$= \int {\rm d}p\, |\widehat{U}|^2\, \delta\left(\omega - \frac{p^2}{m}\right) = \int {\rm d}\varepsilon \sqrt{\frac{2m}{\varepsilon}}\; |\widehat{U}|^2\, \delta(\omega - 2\varepsilon) \sim |\widehat{U}|^2 \sqrt{\frac{m}{\omega}}\,.$$

4. *(10 marks)* Evaluate the matrix elements

$$\langle 0|\, \hat{a}_p\, \hat{a}_q\; \hat{a}^\star_k\, \hat{a}^\star_l\, |0\rangle \quad \text{and} \quad \langle 0|\, \hat{a}_p\, \hat{a}_q\, \hat{a}_l\, \hat{a}^\star_k\, \hat{a}^\star_l\, |0\rangle\,, \tag{4.1}$$

for the creation and annihilation operators for continuum normalized fermions, that satisfy $\{\hat{a}_p\,, \hat{a}^\star_q\} = \delta(p-q)$. Here $|0\rangle$ is the usual no-particle state defined by $\hat{a}_p|0\rangle = 0$.

Any expectation value $\langle\psi|...|\psi\rangle$ involving an unequal number of $\hat{a}_p$'s and $\hat{a}^\star_q$'s must vanish because it changes the total number of particles in the initial state and so gives a result that is orthogonal to the initial state. So the second example in the question is

$$\langle 0|\, \hat{a}_p\, \hat{a}_q\, \hat{a}_l\, \hat{a}^\star_k\, \hat{a}^\star_l\, |0\rangle = 0.$$

In the first example repeatedly use $\hat{a}_p\hat{a}^\star_q = -\hat{a}^\star_q\hat{a}_p + \delta(p-q)$, together with $\hat{a}_p|0\rangle = 0$ and $\langle 0|\hat{a}^\star_q = 0$.

$$\begin{aligned}\langle 0|\, \hat{a}_p\, \hat{a}_q\ \hat{a}^\star_k\, \hat{a}^\star_l\, |0\rangle &= -\langle 0|\, \hat{a}_p\, \hat{a}^\star_k\, \hat{a}_q \hat{a}^\star_l\, |0\rangle + \delta(k-q)\langle 0|\, \hat{a}_p\, \hat{a}^\star_l\, |0\rangle \\ &= -\delta(p-k)\,\delta(q-l) + \delta(k-q)\,\delta(p-l)\,. \end{aligned} \tag{4.2}$$