Rotational Kinetic Energy and Angular Momentum

Concepts: Rotational kinetic energy

Angular momentum

Conservation of angular momentum

Chapter 13: 13.7-13.10

Rotational Kinetic Energy

- Just as there is kinetic energy associated with the motion of an object’s centre of mass (i.e. translational motion), there is kinetic energy associated with rotation
- An object can have both kinds of kinetic energy

- Recall: kinetic energy due to translation is $K = \frac{1}{2} mv^2$

- Kinetic energy due to rotation:

$$K_{rot} = \tfrac{1}{2} I \omega^2$$

- Kinetic energy of a body undergoing rotational and translational motion

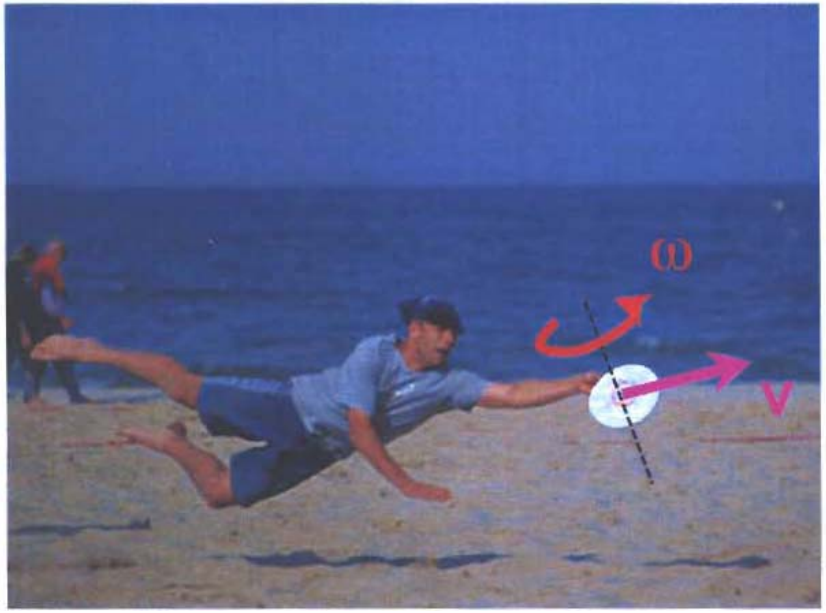
$$K = \frac{1}{2} mv^2 + \frac{1}{2} I\omega^2$$

Examples

- Kinetic energy of a spinning top
  - rotation axis is fixed

$$K_{rot} = \tfrac{1}{2} I \omega^2$$

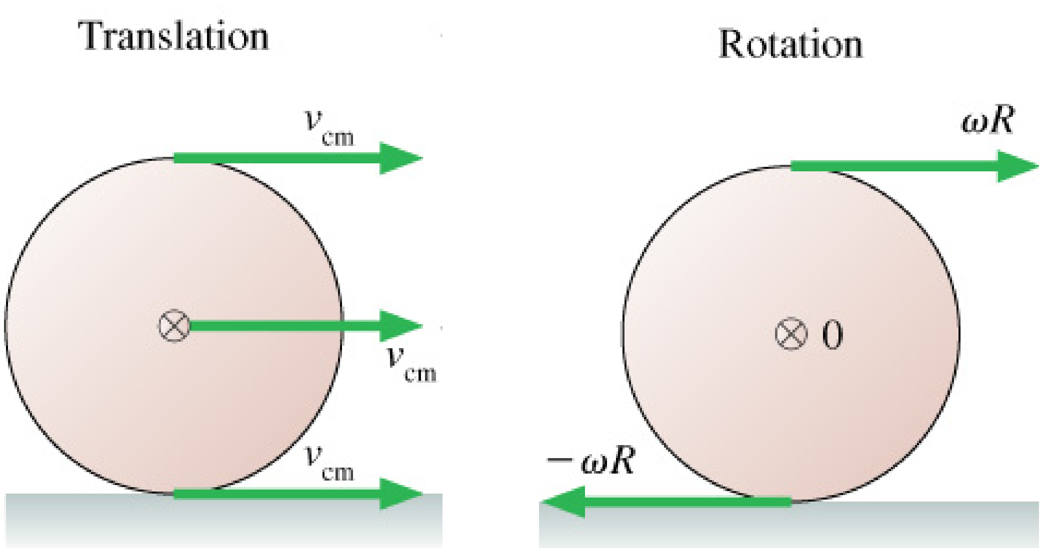
- Kinetic energy of a flying Frisbee



$$K = \tfrac{1}{2} m v_{CM}^2 + \tfrac{1}{2} I_{CM} \omega^2$$

Rolling Motion

$$K = \frac{1}{2} m v_{cm}{}^{2} + \frac{1}{2} I_{cm} \omega^{2}$$



Conservation of Energy

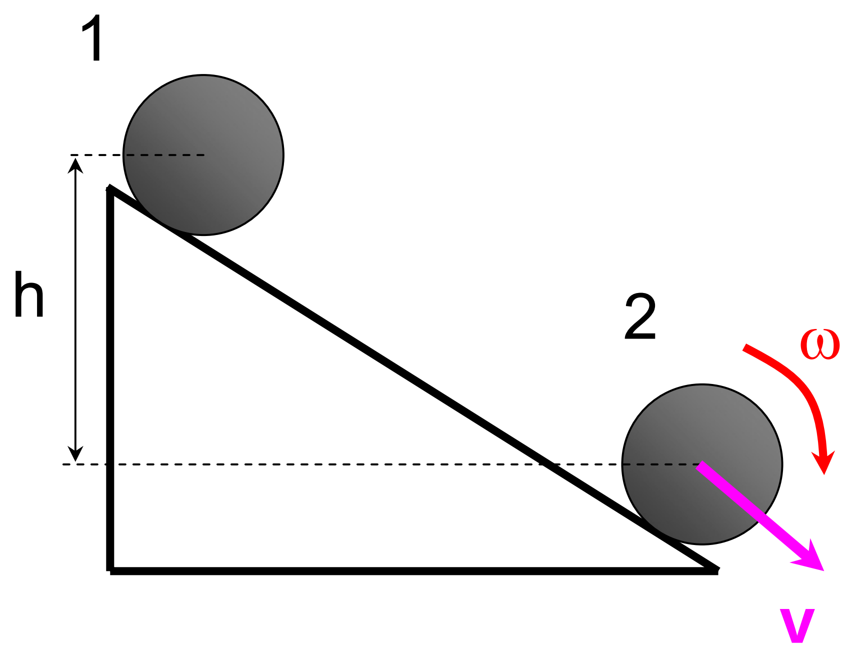
Still applies with rotational motion:

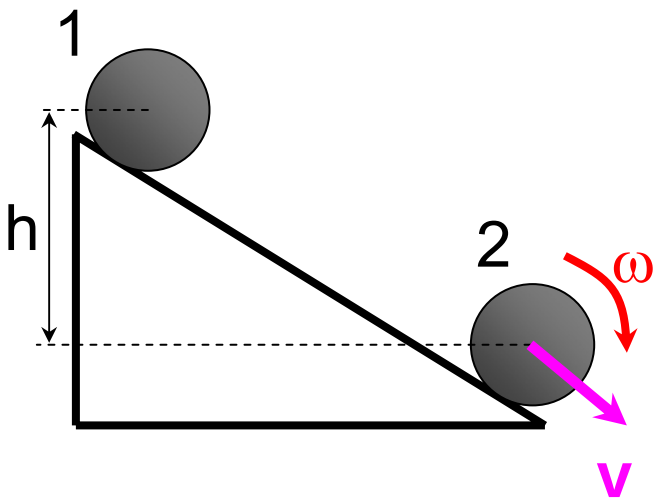
$$E_{mech} = K_i + U_i = K_f + U_f$$

Kinetic energy, K, can be either translational or rotational, or both.

Example

A ball of mass, M, and radius, R, initially at rest at height, h, on a ramp rolls without slipping along the ramp. Find its angular and linear velocity at the bottom of the ramp.





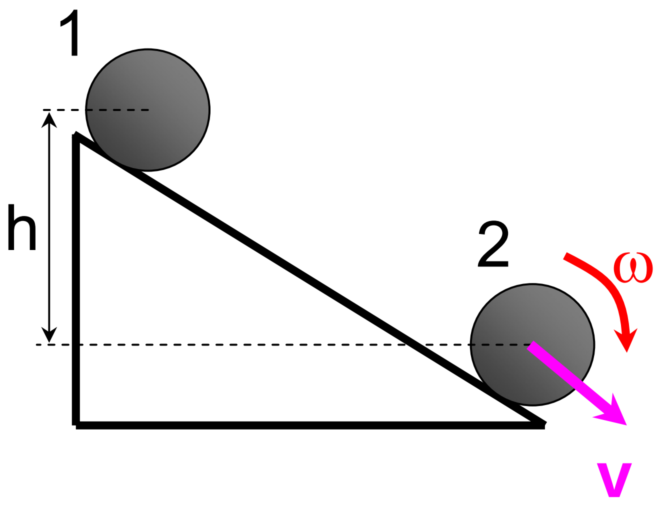
Known: mass, M

radius, R

$I_{sphere} = 2/5\ MR^2$

Find: v, ω

Apply conservation of energy: $E_1 = E_2$



Known: mass, M

radius, R

$I_{sphere} = 2/5\ MR^2$

Find: v, ω

Apply conservation of energy: $E_1 = E_2$

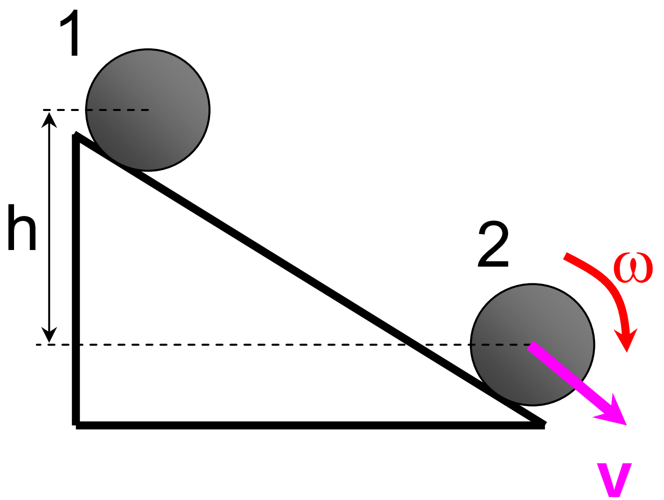
Position 1: $K_1 = 0$ and $U_1 = Mgh$

Position 2: $K_2 = K_{trans} + K_{rot} =$

$\rightarrow K_2 = \frac{1}{2} Mv2 + \frac{1}{2} I \omega^2$

$= \frac{1}{2} Mv^2 + \frac{1}{2}(2/5\ MR^2)\omega^2$

$U_2 = 0$ (we set the bottom of the ramp as our origin)



Known: mass, M

radius, R

$I_{sphere} = 2/5\ MR^2$

Find: v, ω

Apply conservation of energy: $E_1 = E_2$

Position 1: $K_1 = 0$ and $U_1 = Mgh$

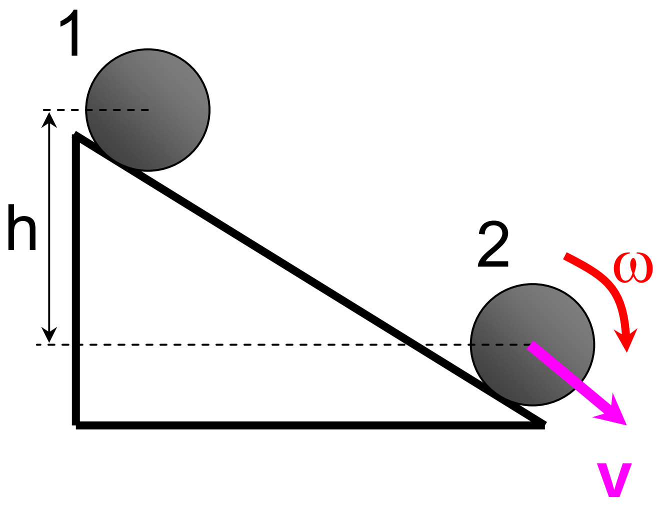
Position 2: $K_2 = K_{trans} + K_{rot} = \frac{1}{2} Mv^2 + \frac{1}{2}(2/5\ MR^2)\omega^2$

$U_2 = 0$ (we set the bottom of the ramp as our origin)

$$K_1 + U_1 = K_2 + U_2$$

$$Mgh = \frac{1}{2} M\, v^2 + \frac{1}{2}\left(\frac{2}{5} MR^2\right)\left(\frac{v}{R}\right)^2$$

$$2Mgh = \left(1 + \frac{2}{5}\right) Mv^2$$



Known: mass, M

radius, R

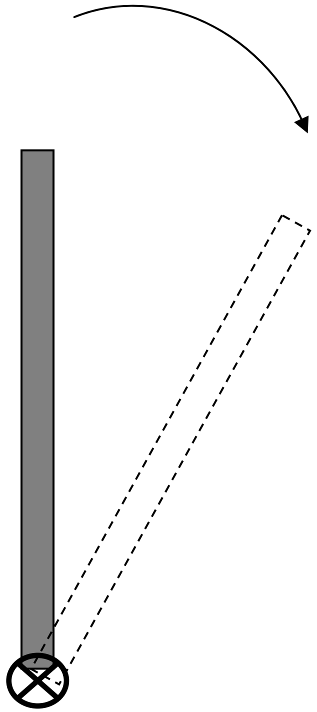
I_{sphere} = 2/5 MR^2

Find: v, ω

Apply conservation of energy: $E_1 = E_2$

$$v^2 = \frac{2gh}{(1+\frac{2}{5})} \xrightarrow{v=\omega R} \omega^2 = \frac{2gh}{R^2(1+\frac{2}{5})}$$

Example



A thin cylindrical rod of length L is allowed to fall over. What is its angular velocity when it hits the ground?

Answer

$$K_i + U_i = K_f + U_f$$

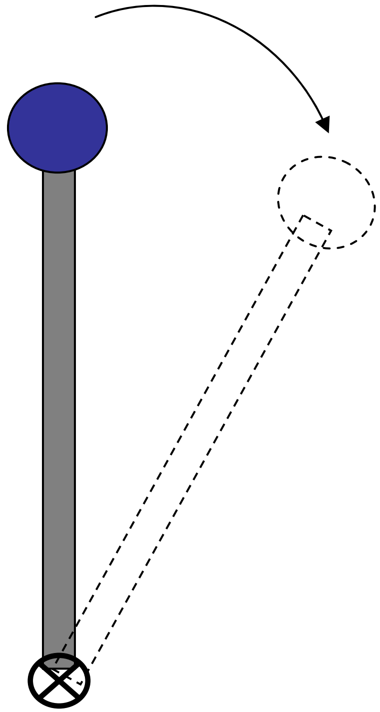
$$U_i = K_f$$

$$0 + Mg\frac{L}{2} = \frac{1}{2} I_{TOT}\omega^2 + 0$$

$$\omega^2 = MgL / I$$

$$\omega^2 = \frac{MgL}{\frac{1}{3}ML^2} \Rightarrow \omega = \sqrt{\frac{3g}{L}}$$

Quick Quiz 27



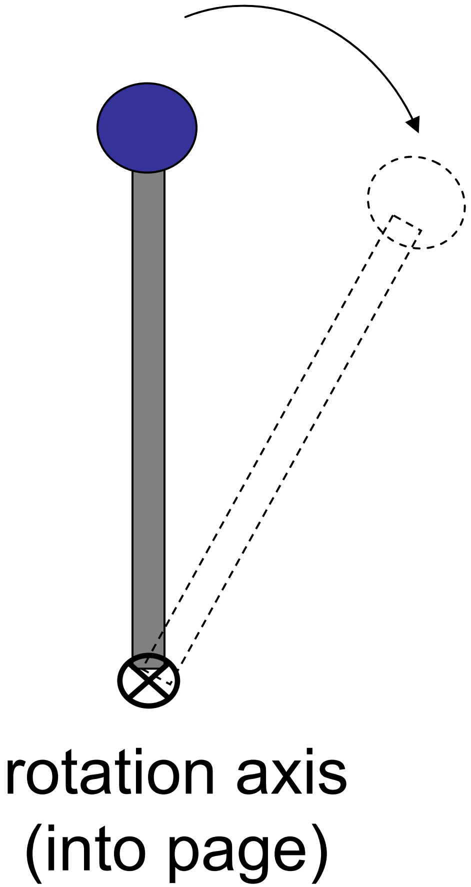
rotation axis
(into page)

A thin cylindrical rod 25 cm long with a mass of 1 kg has a ball with a mass of 0.5 kg and a radius of 1.54 cm attached to one end.

What is the moment of inertia of this object?

A. 0.005 kg m^2
B. 0.05 kg m^2
C. 0.5 kg m^2
D. 5 kg m^2

Quick Quiz 28



The arrangement is originally vertical and stationary, with the ball at the top. The apparatus is free to pivot about the bottom end of the rod. After the ball has fallen through 90 degrees, what is its translational speed?

A. 12 m/s

B. 15 m/s

C. 6 m/s

D. 3 m/s

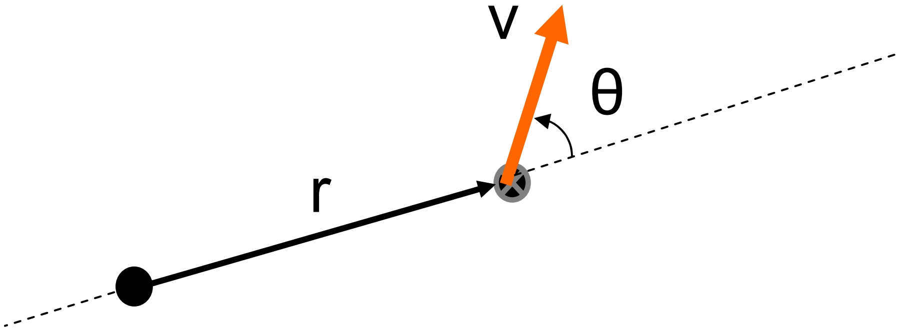
Angular Momentum

- A mass m moving at speed v has a linear momentum of p = mv
- A mass m moving with linear momentum p in a circle of radius r has an ***angular momentum*** of:

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\left|\vec{L}\right| = \left|\vec{r}\right|\left|\vec{p}\right| \sin\theta$$

$$L = mvr\sin\theta$$



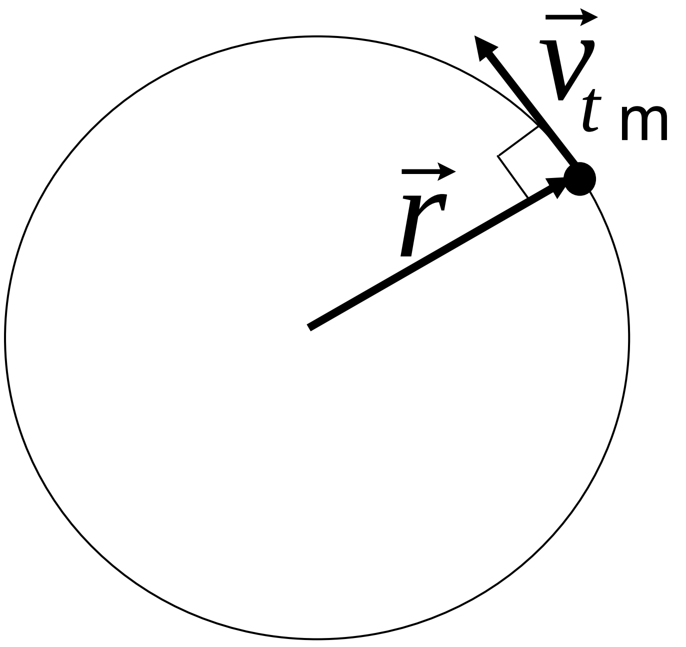
- Consider a point mass, m, moving in a circle of radius r with tangential speed v_t:

$$L = mrv_t \sin(90^o)$$

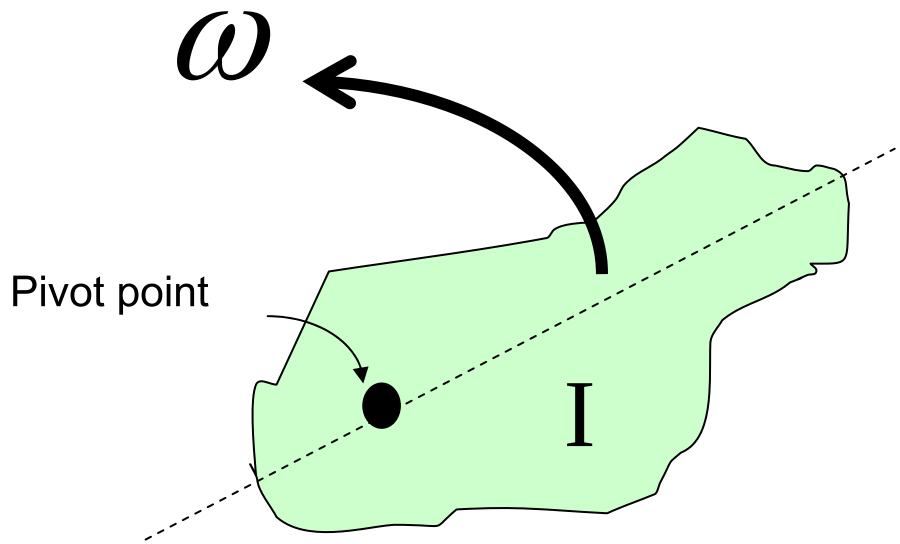
$$L = mrv_t$$

$$L = mr^2\left(\frac{v_t}{r}\right)$$

$$L = I\omega$$



- This result can be shown to apply generally:



$$L = I\,\omega$$ [kg m^2/s]

Newton's Laws for Rotation

- Newton's first law applied to translational motion:
  - ***An object will maintain a constant velocity unless acted upon by some external net force***
- Newton's first law applied to rotational motion:
  - ***An object will maintain a constant angular velocity unless acted upon by some net external torque***

Conservation of Angular Momentum

- Consider the rate of change of angular momentum with time:

$$L = I\omega$$

$$\frac{dL}{dt} = I\frac{d\omega}{dt}$$

$$\frac{dL}{dt} = I\alpha$$

$$\frac{dL}{dt} = \tau$$

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

Torque is the rate of change of angular momentum with time.

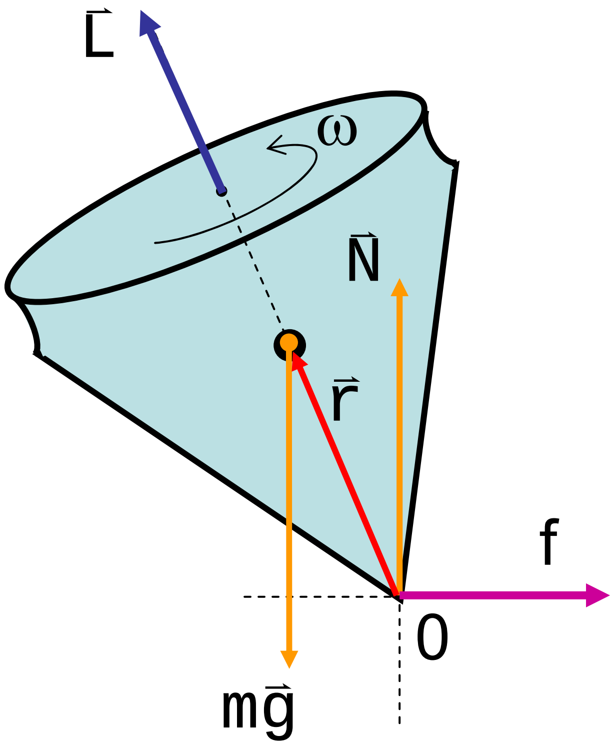
If there is no torque, L doesn't change!

A spinning top that is not upright **precesses**.

Choose the origin at the contact point. N, f_x, f_y give no torque

$\vec{\tau} = \vec{r} \times m\vec{g}$ is out of the page, $\perp \vec{L}$

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$ is out of the page



Hanging bike wheel demo

- The previous relationship should look familiar:

$$F = \frac{dp}{dt} = \frac{d(mv)}{dt} = m\frac{dv}{dt} = ma$$

- **Force** is the rate of change of **linear** momentum

- If there is no net torque on an object, then dL/dt = 0 and angular momentum is conserved

Linear momentum:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

$$\vec{F} = 0 \Rightarrow \vec{\mathrm{p}} \text{ conserved}$$

The linear momentum of an isolated system, on which the net force is zero, is conserved.

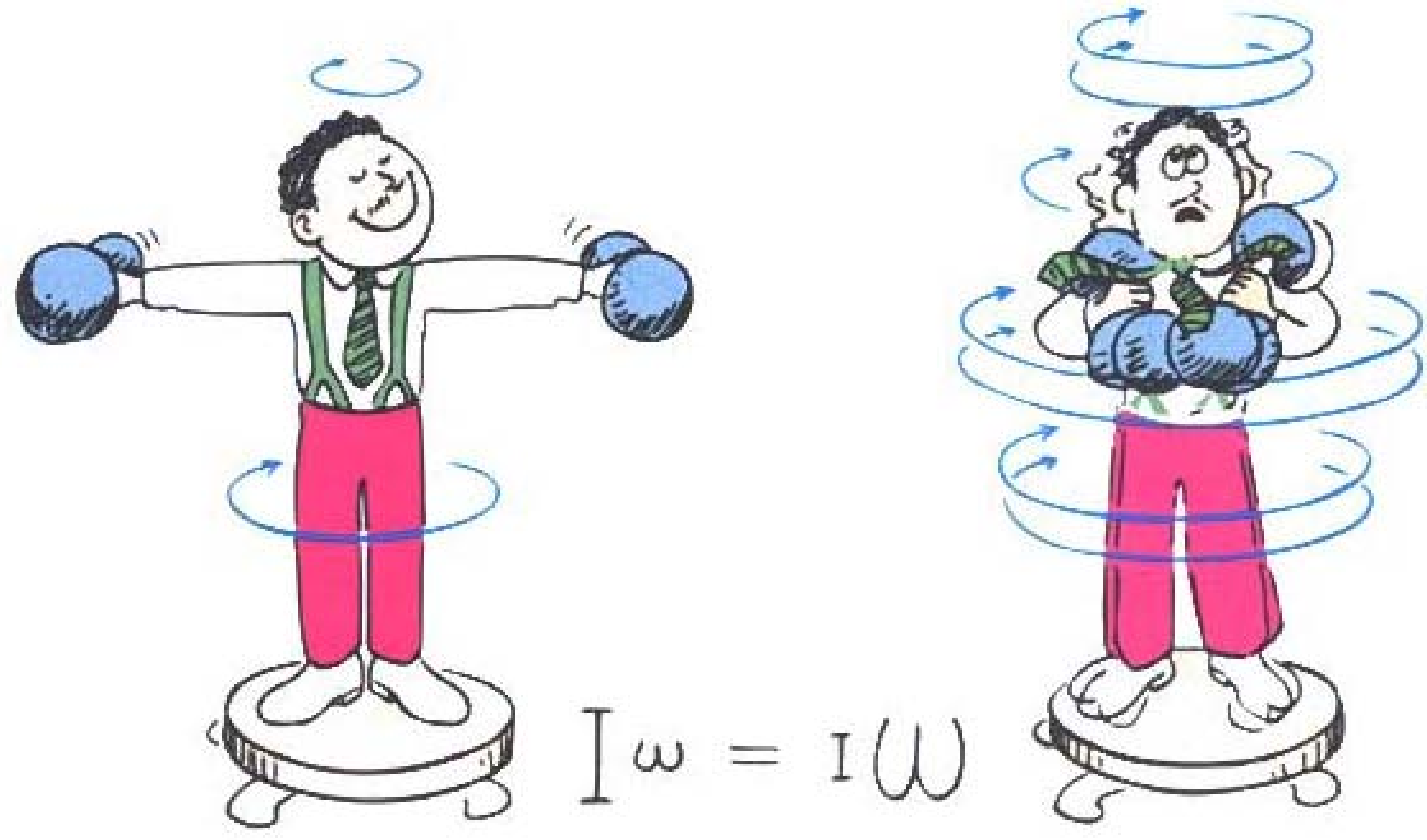
Angular momentum:

$$\vec{\tau} = \frac{d\vec{L}}{dt}$$

$$\vec{\tau} = 0 \Rightarrow \vec{\mathrm{L}} \text{ conserved}$$

The angular momentum of an isolated system, on which the net torque is zero, is conserved.

Conservation of Angular Momentum



Example: Spinning Skater

- The moment of inertia of the skater is changing as she pulls her arms and legs in.
- The external torque on her is zero.
- Her angular momentum is conserved. (neglecting friction torque)

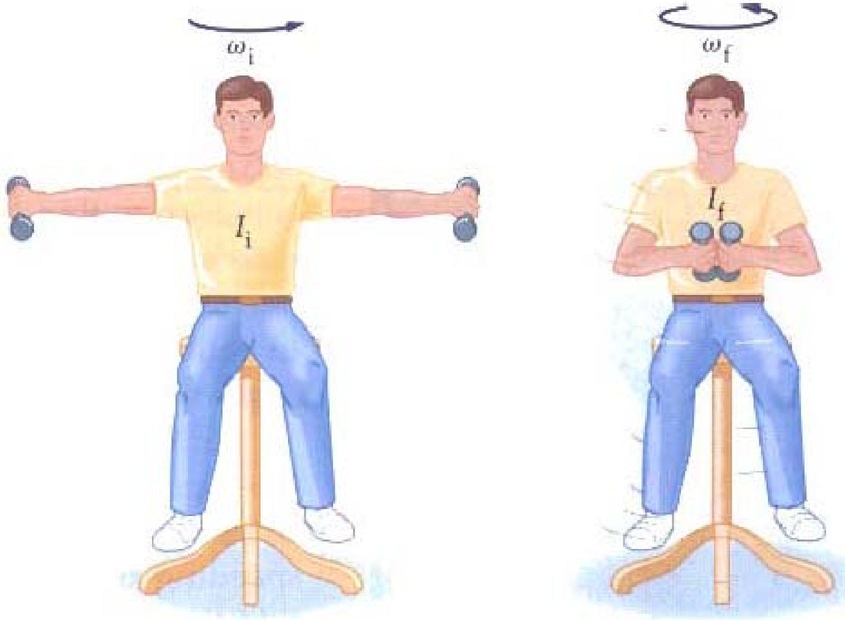
$$L_i = L_f$$

$$I_i\omega_i = I_f\omega_f$$

$$\omega_f = \frac{I_i\omega_i}{I_f}$$

Quick Quiz 29

A student sits on a stool holding a dumbbell in each hand. Initially, the student holds his arms outstretched and spins about the axis of the stool with an angular speed of 3.75 rad/s. The moment of inertia in this case is 3.20 kg m^2. While still spinning, the student pulls his arms in to his chest, reducing his moment of inertia to1.60 kg m^2. What is the student's new angular speed? (*ignore friction*)



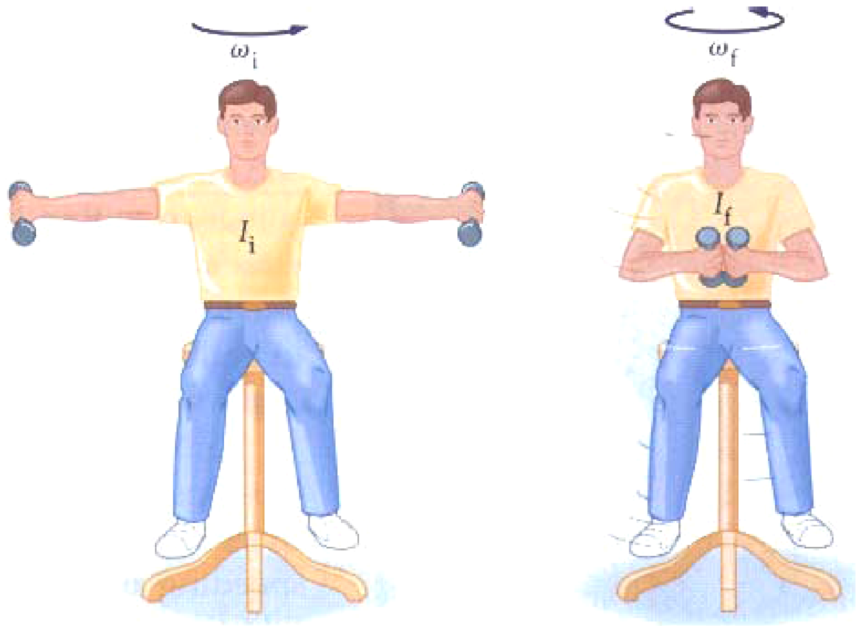
A. 3.75 rad/s

B. 5.0 rad/s

C. 6.5 rad/s

D. 7.5 rad/s

Answer

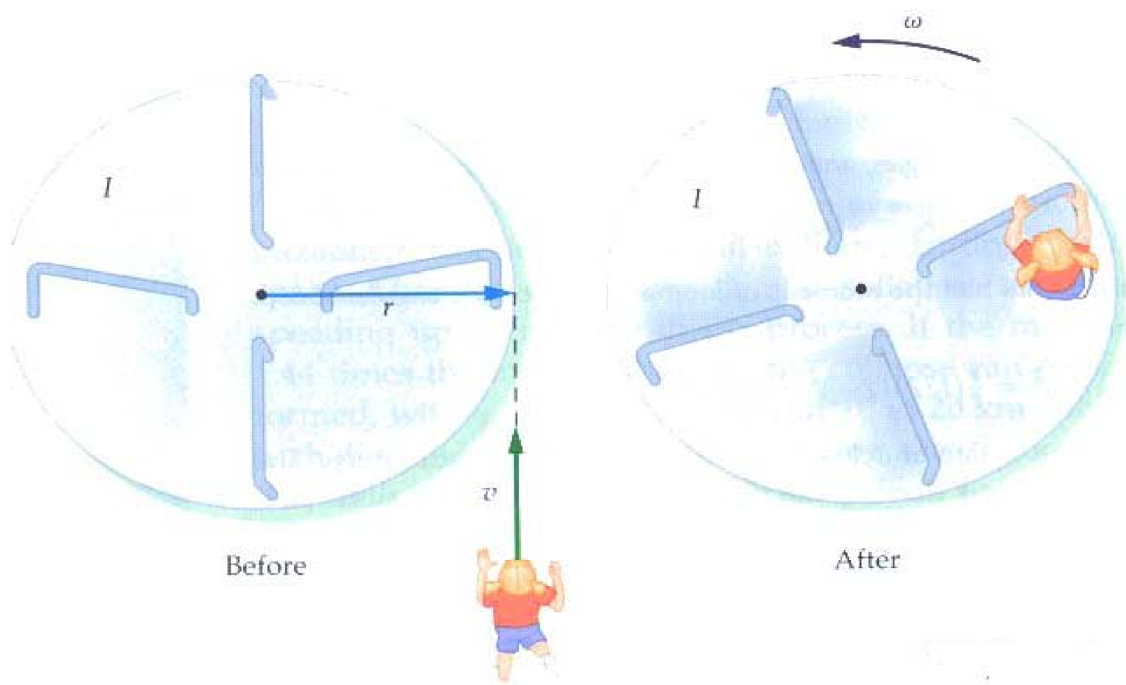


$$I_1\omega_1 = I_2\omega_2$$

$$\omega_2 = \frac{I_1}{I_2}\omega_1 = \frac{3.2}{1.6}(3.75) = \boxed{7.5\text{rad/s}}$$

Example: Jump on the Merry-go-round

A 30 kg child runs with a speed of 3 m/s tangential to the rim of a stationary merry–go–round. The merry–go–round has a moment of inertia of 500 kg m^2 and a radius of 2 m. When the child jumps on the merry–go–round, the entire system begins to rotate. What is the angular speed of the system?



- What’s happening here?
  - initially, the merry-go-round has no momentum—it is at rest
  - the child has constant angular momentum, (r x p), calculated about the unmoving rotation axis of the merry-go-round
  - After the child is connected to the merry-go-round, the total initial angular momentum is shared by the system as a whole

- Conserve angular momentum

$$L_{child} = L_{child+mgr}$$

$$m_{girl} v_t r = I_{child+mgr} \omega$$

$$m_{girl} v_t r = (I_{\mathrm{mgr}} + I_{girl}) \omega$$

$$m_{girl} v_t r = (I_{mgr} + m_{girl} r^2) \omega$$

$$\omega = \frac{m_{girl} v_t r}{(I_{mgr} + m_{girl} r^2)}$$

$$\omega = \frac{(30kg)(3m/s)(2m)}{(500kgm^2 + (30kg)(2m)^2)} = \frac{180}{620} \cong \boxed{0.3\mathrm{rad/s}}$$

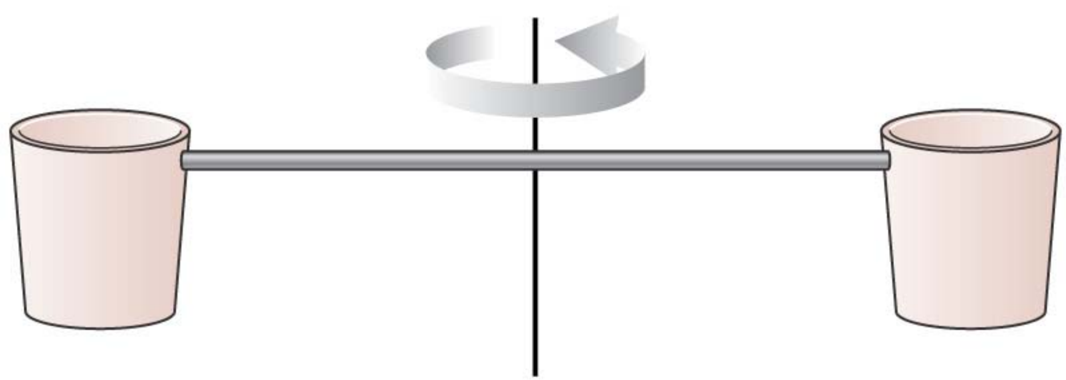
Quick Quiz 30

Two buckets spin around a horizontal circle with constant ω on frictionless bearings. Suddenly, it starts to rain. As a result:

(a) The buckets continue to rotate with constant ω because the rain falls vertically but the buckets rotate in a horizontal plane

(b)The buckets slow down because conservation of angular momentum of the buckets and water is conserved

(c) The buckets speed up because the potential energy of the rain is converted to kinetic energy



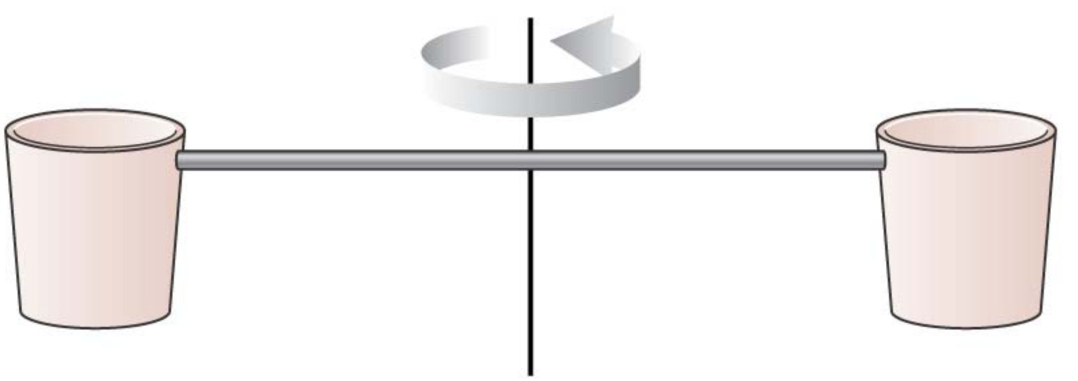
Quick Quiz 31

Two buckets spin around a horizontal circle with constant ω on frictionless bearings. Both buckets have a leak that allows water to slowly drip out of the bottom. As a result:

(a) The buckets continue to rotate with constant ω because the drips falls vertically but the buckets rotate in a horizontal plane

(b)The buckets slow down because conservation of angular momentum of the buckets and water is conserved

(c) The buckets speed up because the potential energy of the rain is converted to kinetic energy



Translational vs. Rotational Motion

Translational	Rotational
Mass	**Moment of inertia**
Force	**Torque**
Linear acceleration	**Angular acceleration**
Linear velocity	**Angular velocity**
Position	**Angular position**
Newton's second law	**Newton's second law for rotational motion**
Translational kinetic energy	**Rotational kinetic energy**
Conservation of energy	**Conservation of energy**
Linear momentum	**Angular momentum**
Conservation of momentum	**Conservation of angular momentum**