Superposition of Waves

1) Identical waves in opposite directions:
   "standing waves"

2) 2 waves at slightly different frequencies:
   "beats"

3) 2 identical waves, but not in phase:
   "interference"

Principle of Superposition

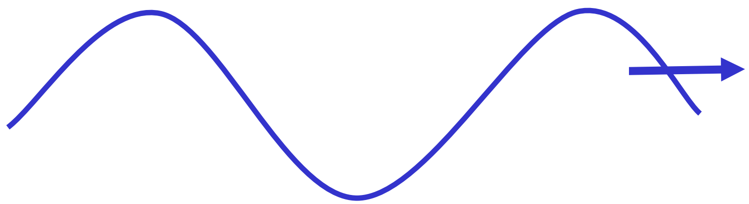
2 Waves In The Same Medium:

The observed displacement $y(x,t)$ is the sum of the individual displacements:

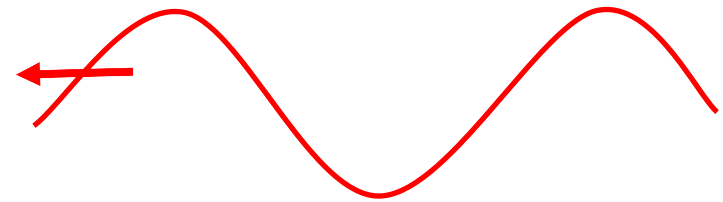
$$y_1(x,t) + y_2(x,t) = y(x,t)$$

(for a "linear medium")

Sine Waves In Opposite Directions:



$y_1 = A_o \sin(kx - \omega t)$



$y_2 = A_o \sin(kx + \omega t)$

Total displacement: $y(x,t) = y_1 + y_2$

Trigonometry: $\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$$y(x,t) = 2A_0 \sin kx \cos \omega t$$

$$y(x,t) = (2A_0 \sin kx)\cos \omega t$$

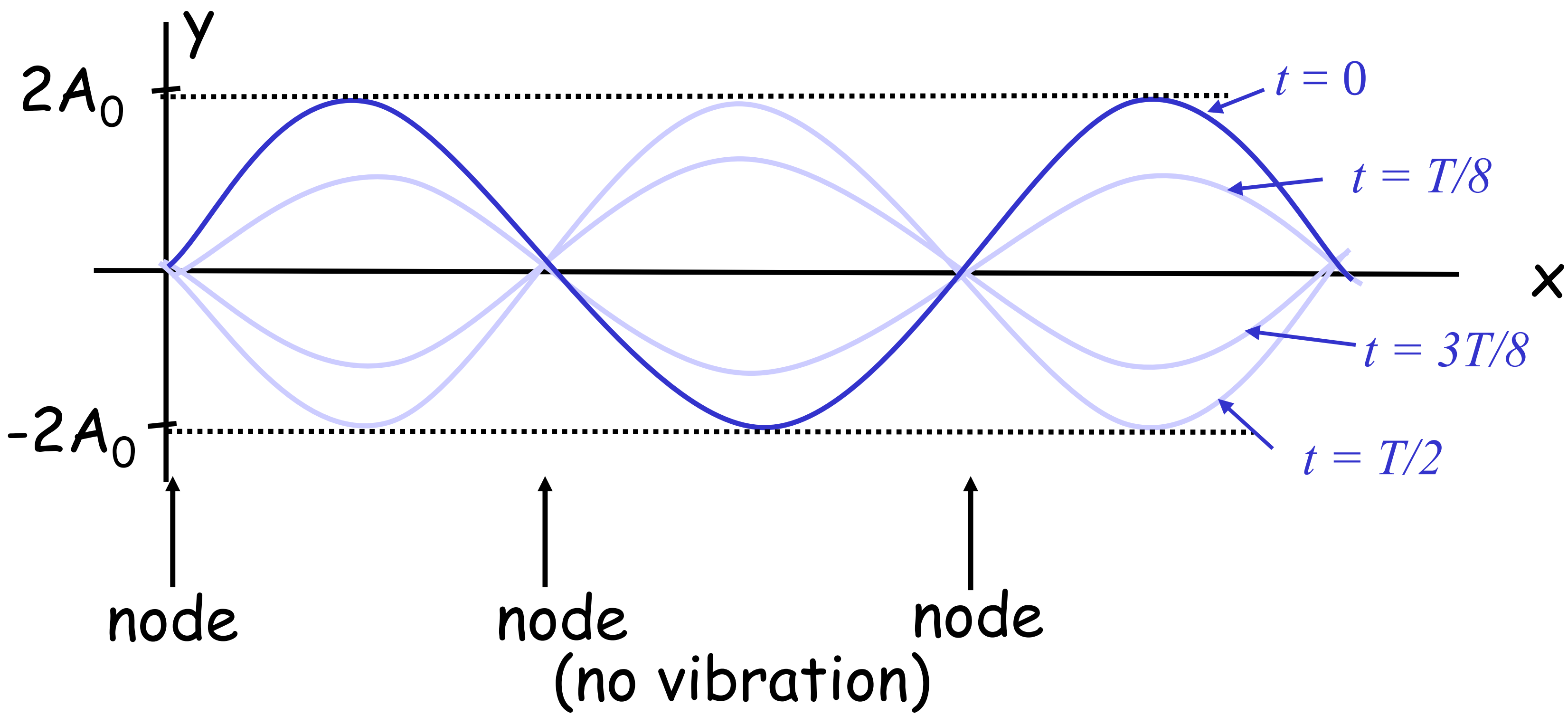
At a given value of x, this is a constant

Each point oscillates with frequency ω, and amplitude $2A_0\sin(kx)$

But the constant amplitude at a given point can be zero!

Standing Waves

The particle motions are simple harmonic oscillations which are all in phase (or ½ cycle out of phase) with each other, but with different amplitudes.



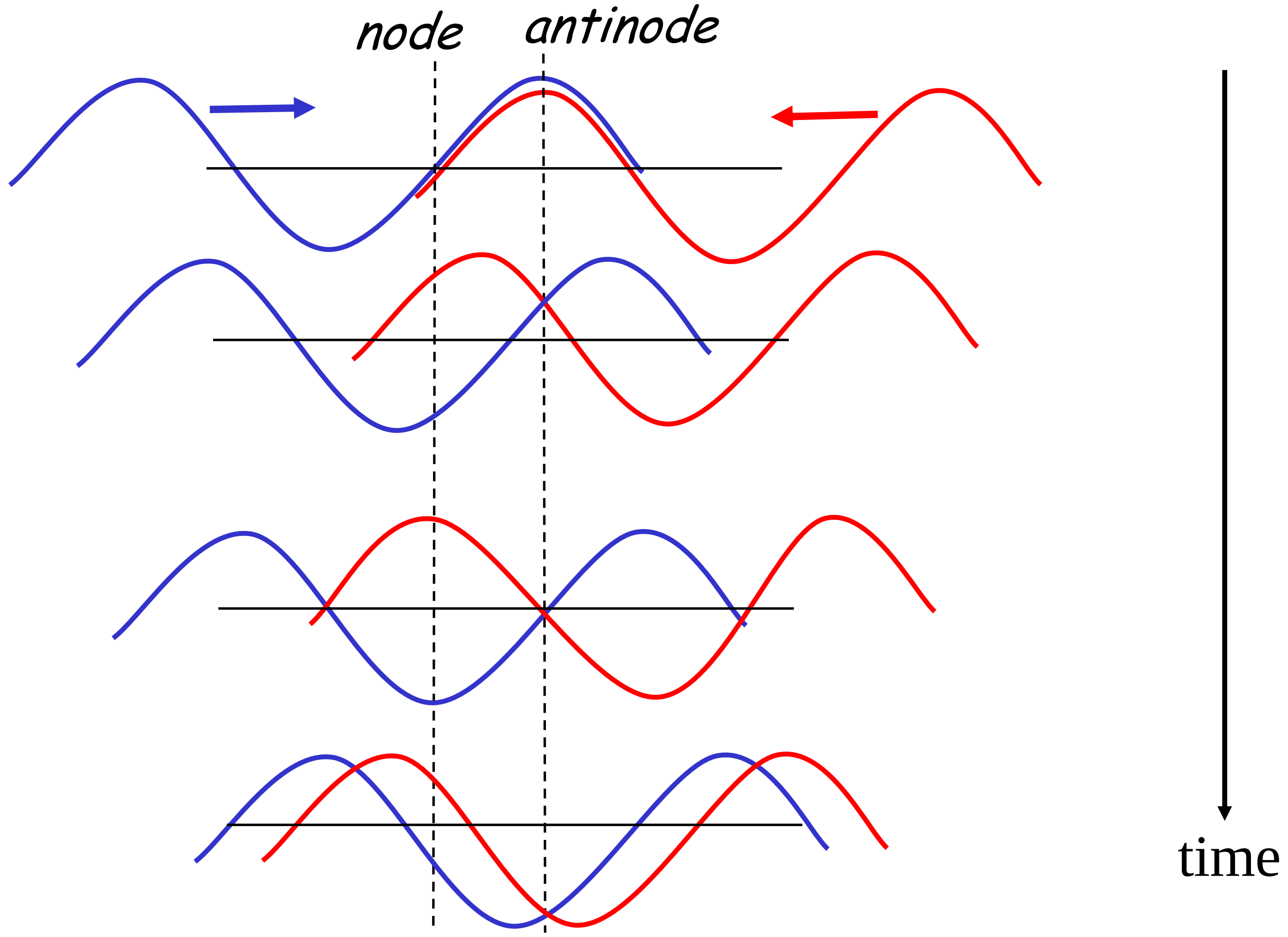
Nodes are positions where the amplitude, $2A_0\sin(kx) = 0$:

this happens when:

$$kx = 0 \qquad (x = 0)$$
$$kx = \pi \qquad (x = \lambda/2)$$
$$kx = 2\pi \qquad (x = \lambda)$$
etc.

i.e., **Nodes are ½ wavelength apart.**

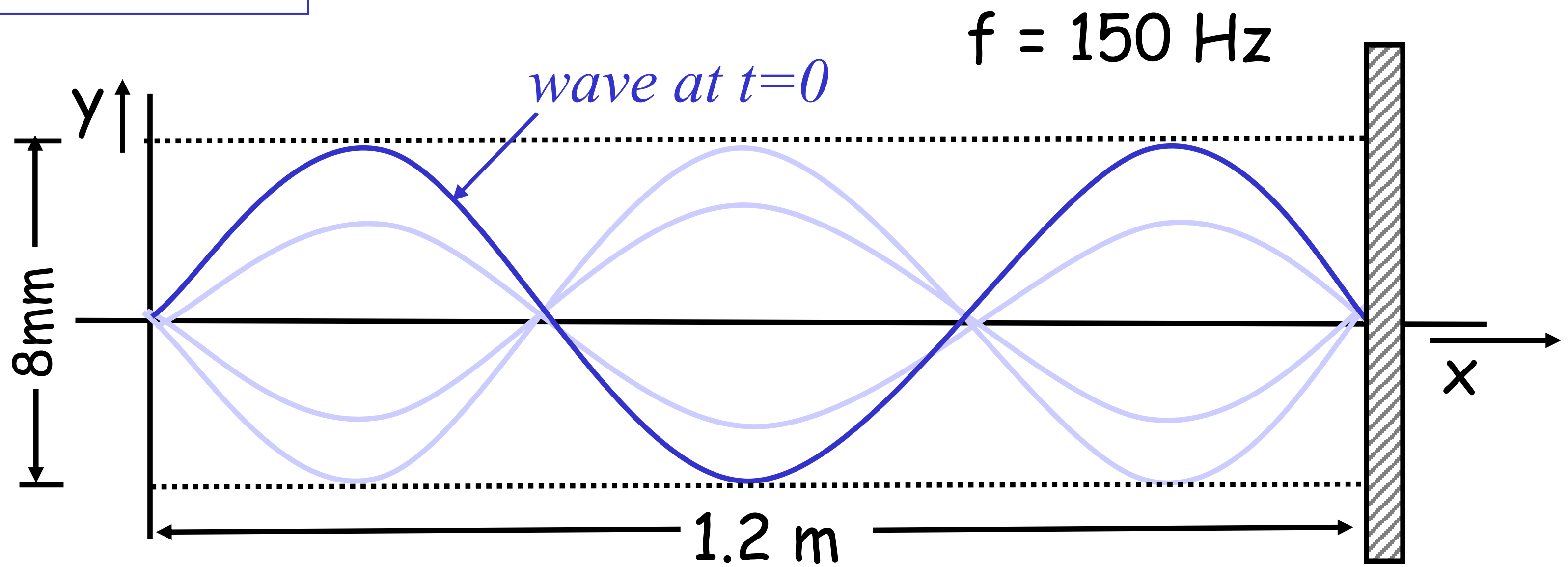
Antinodes (maximum amplitude) are halfway between nodes.



Antinodes form where the waves always arrive in phase ("constructive interference"); nodes form at locations where the waves are 180° (½ cycle) out of phase ("destructive interference").

Quick Quiz

$$y(x,t) = (2A_0 \sin kx)\cos \omega t$$



What is y(x,t) for the standing wave?

a) $4\sin(5x)\cos(30\pi t)\ [mm]$

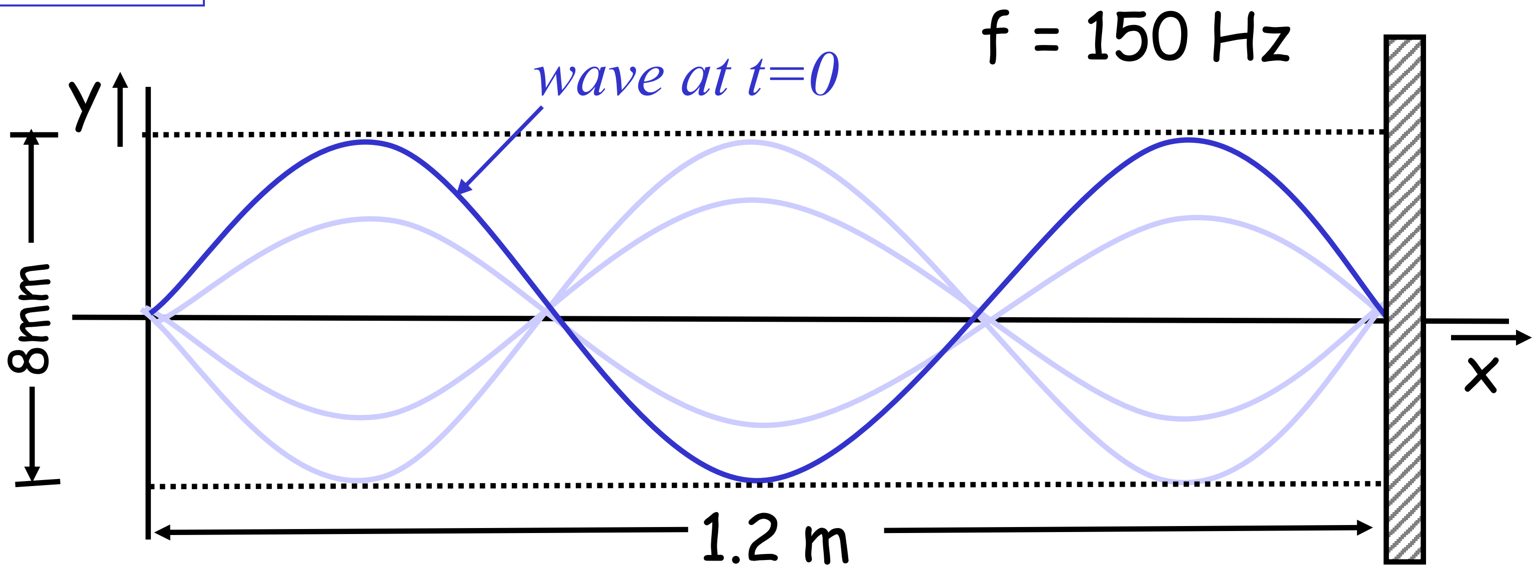
b) $8\sin(0.3x)\cos(150t)\ [mm]$

c) $4\sin(2.5\pi x)\cos(300\pi t)\ [mm]$

d) $8\sin(2.5x)\cos(300t)\ [mm]$

Example

$$y(x,t) = (2A_0 \sin kx)\cos \omega t$$

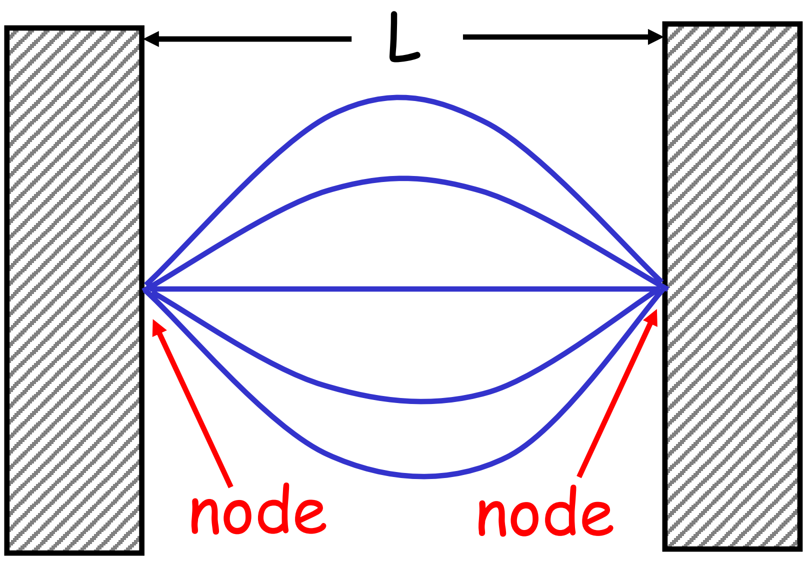


What is $y_1(x,t)$ and $y_2(x,t)$ for the two <u>traveling</u> waves which would produce this standing wave.

Practical Setup: Fix the ends, use reflections.

We can think of travelling waves reflecting back and forth from the boundaries, and creating a standing wave. The resulting standing wave must have a node at each fixed end. Only certain wavelengths can meet this condition, so only certain particular frequencies of standing wave will be possible.

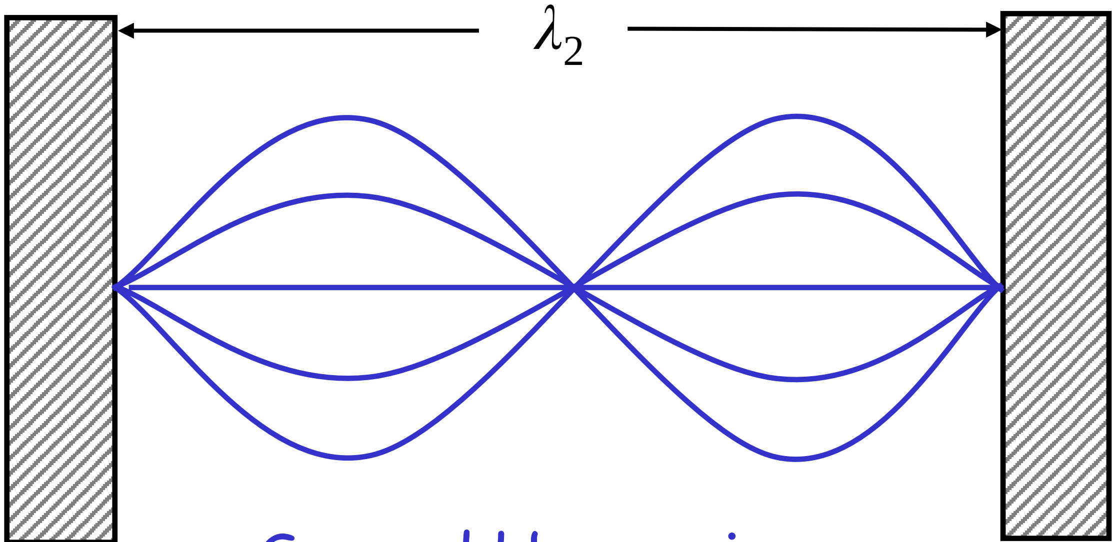
example:



$$\tfrac{1}{2}\lambda_1 = L \Rightarrow \qquad \lambda_1 = 2L$$

$$f_1 = \frac{v}{2L}$$

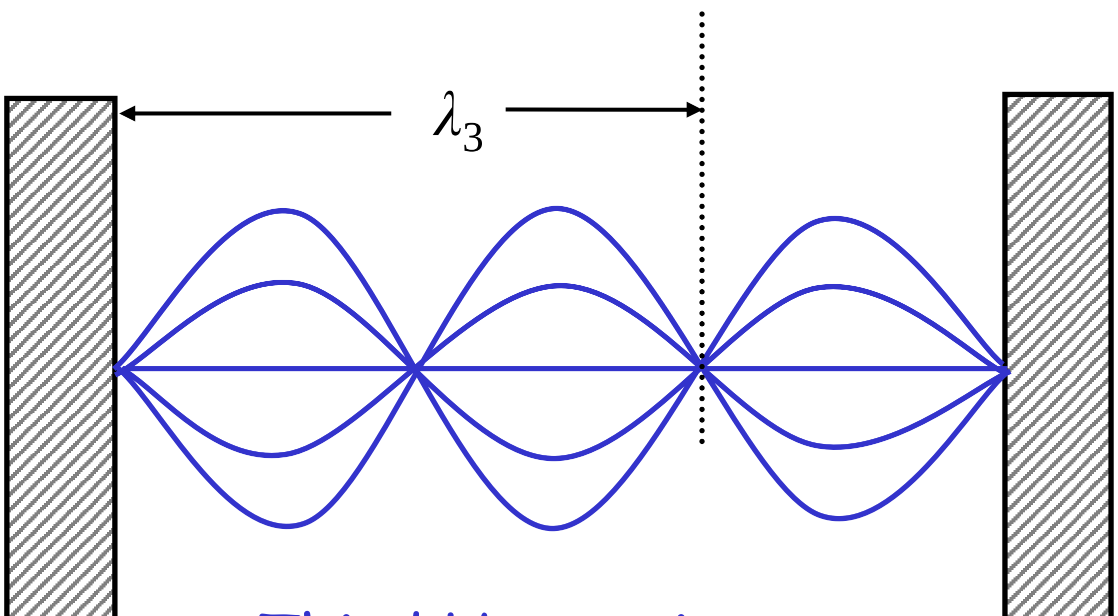
("fundamental mode")



Second Harmonic

$$\lambda_2 = L$$

$$f_2 = \frac{v}{L} = 2f_1$$



Third Harmonic . . . .

$$\tfrac{3}{2}\lambda_3 = L$$

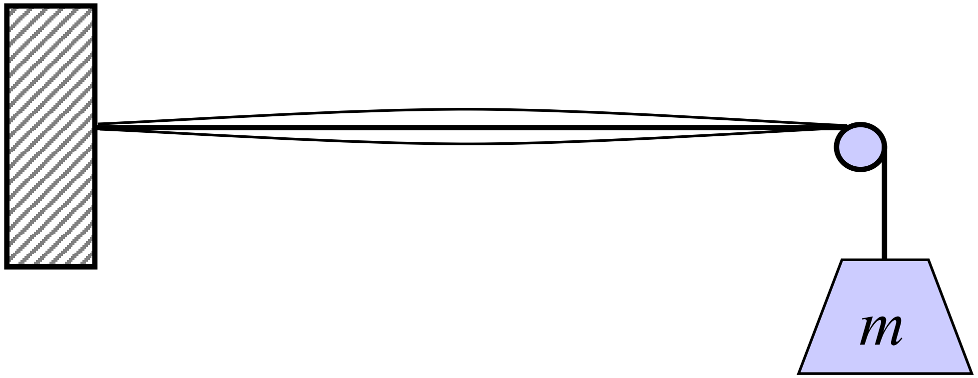
$$\lambda_3 = \tfrac{2}{3}L$$

$$f_3 = \frac{v}{\lambda_3} = 3\frac{v}{2L}$$

$$= 3f_1$$

Quick Quiz

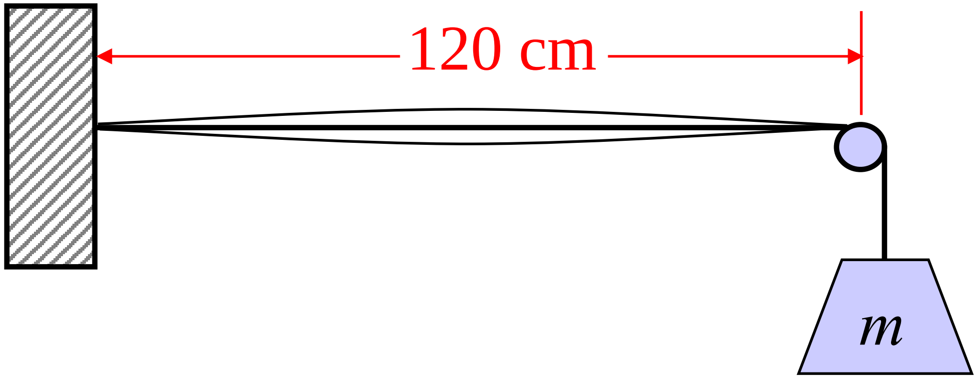
$$f\lambda = v = \sqrt{\frac{F_T}{\mu}}$$



When the mass m is doubled, what happens to the wavelength, and the frequency of the fundamental standing-wave mode?

a) Both wavelength and frequency stay the same.
b) Wavelength decreases. Frequency increases.
c) Wavelength stays the same. Frequency decreases.
d) Wavelength increases. Frequency decreases.
e) Wavelength stays the same. Frequency increases.

Example



a) $m = 150\text{g}$, $f_1 = 30$ Hz. Find μ (mass per unit length)

b) Find m needed to give $f_2 = 30$ Hz

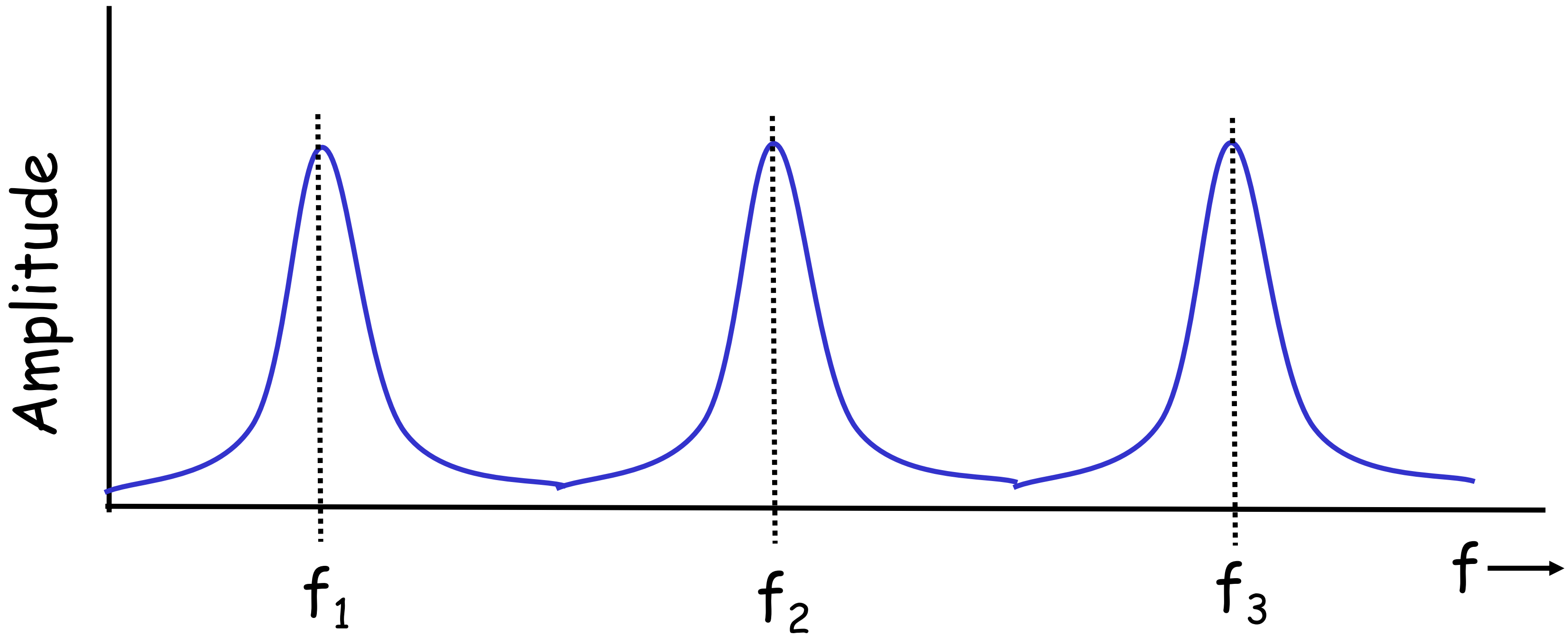
Resonance

What happens if you *force the string to vibrate* at a frequency which is *not* one of the standing-wave frequencies?

Let "natural" frequency of object = f_0
External periodic force frequency = f(ext)

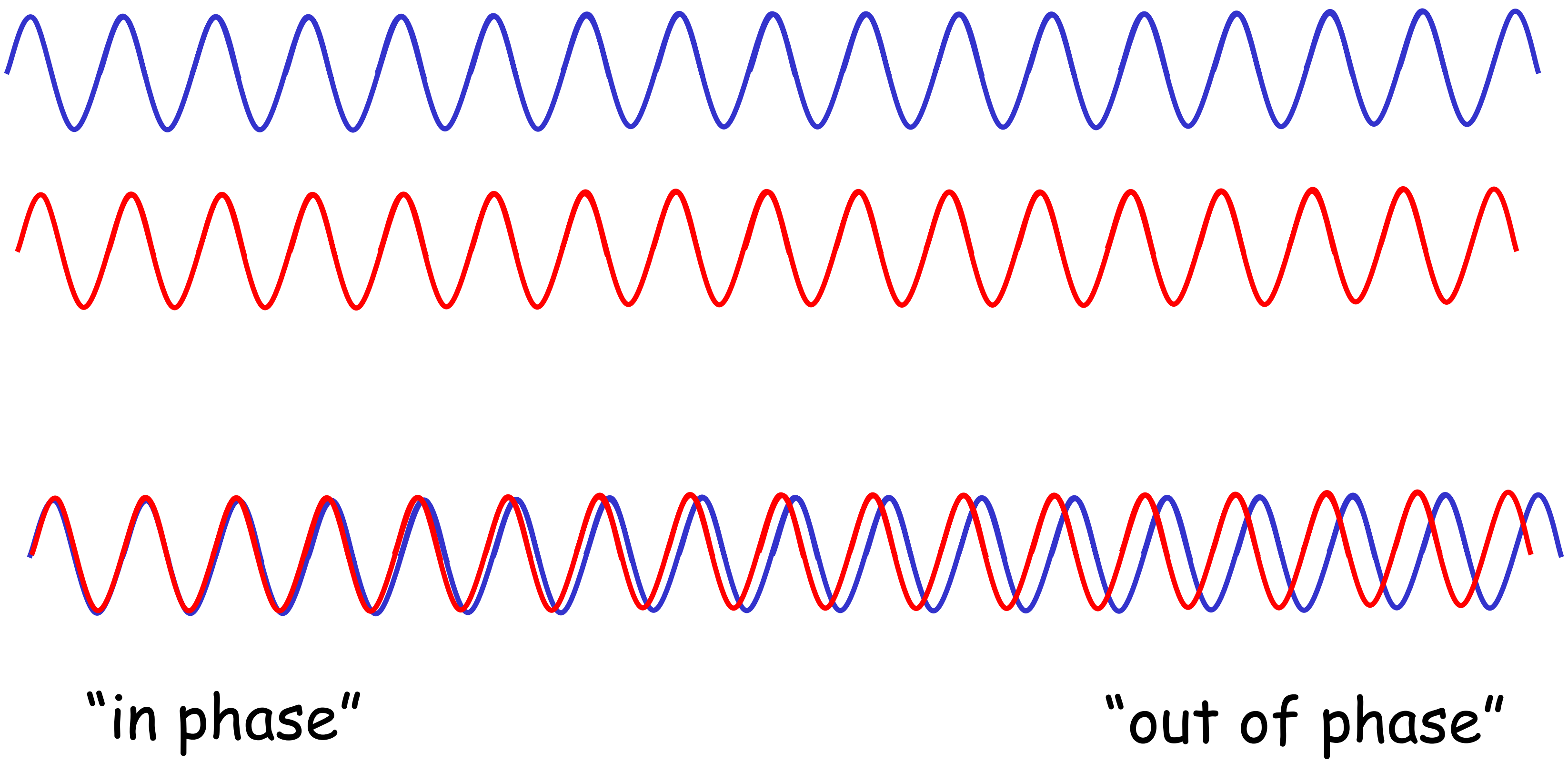
Result: the system *will* oscillate at f(ext)
but at small amplitude

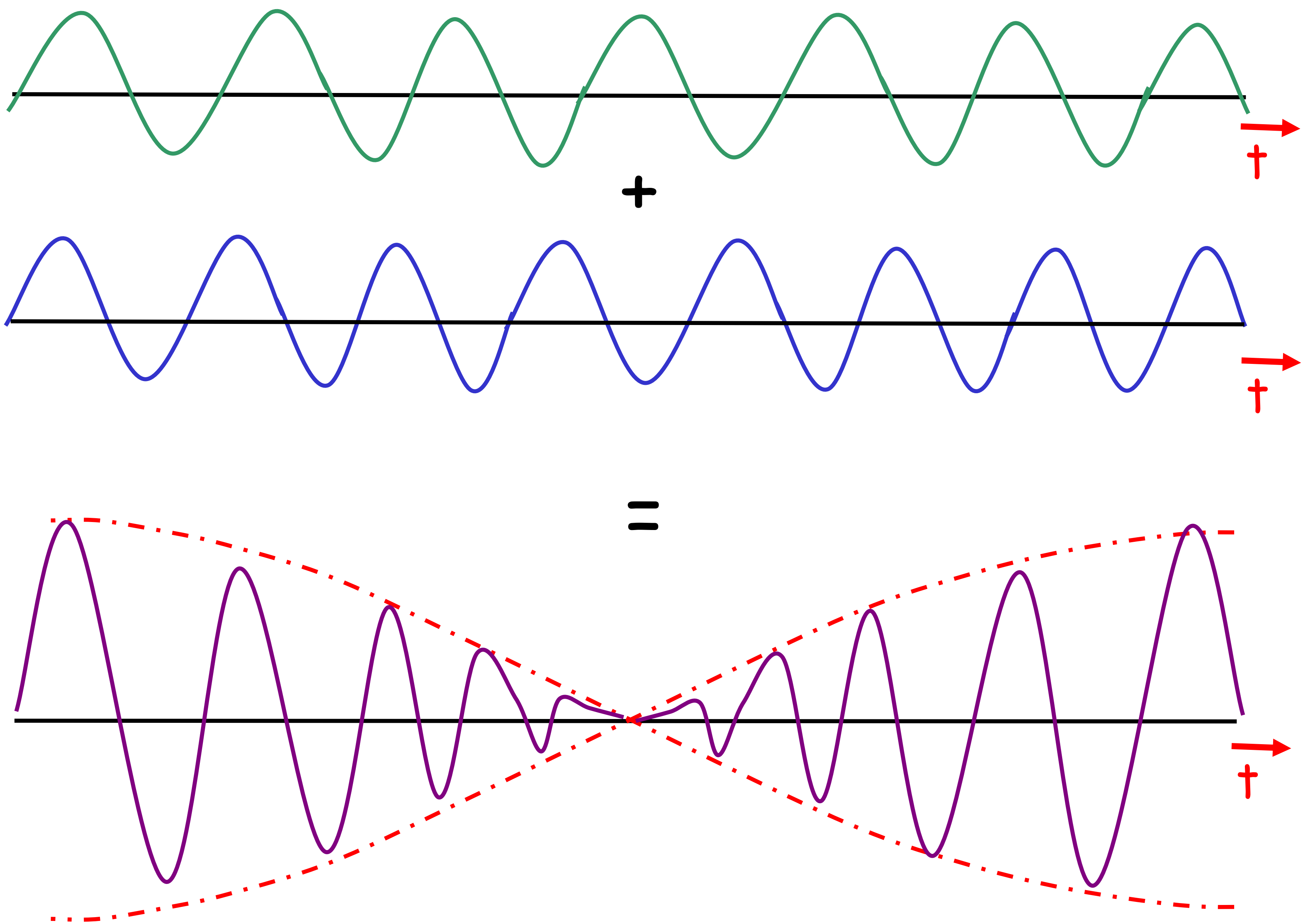
The amplitude will increase dramatically whenever f(ext) is close to one of the string's natural harmonics



The natural harmonics of the system are also called its "resonant frequencies".

Now consider adding up 2 waves with slightly *different* frequencies ω_1, ω_2





+
=
t
t
t

Same amplitude, different frequencies at x=0:

$$y_1 = A\cos(\omega_1 t) \qquad\qquad y_2 = A\cos(\omega_2 t)$$

Trigonometry:

cos a + cos b = 2 cos [(a-b)/2] cos [(a+b)/2]

So result:

$$y = y_1 + y_2$$

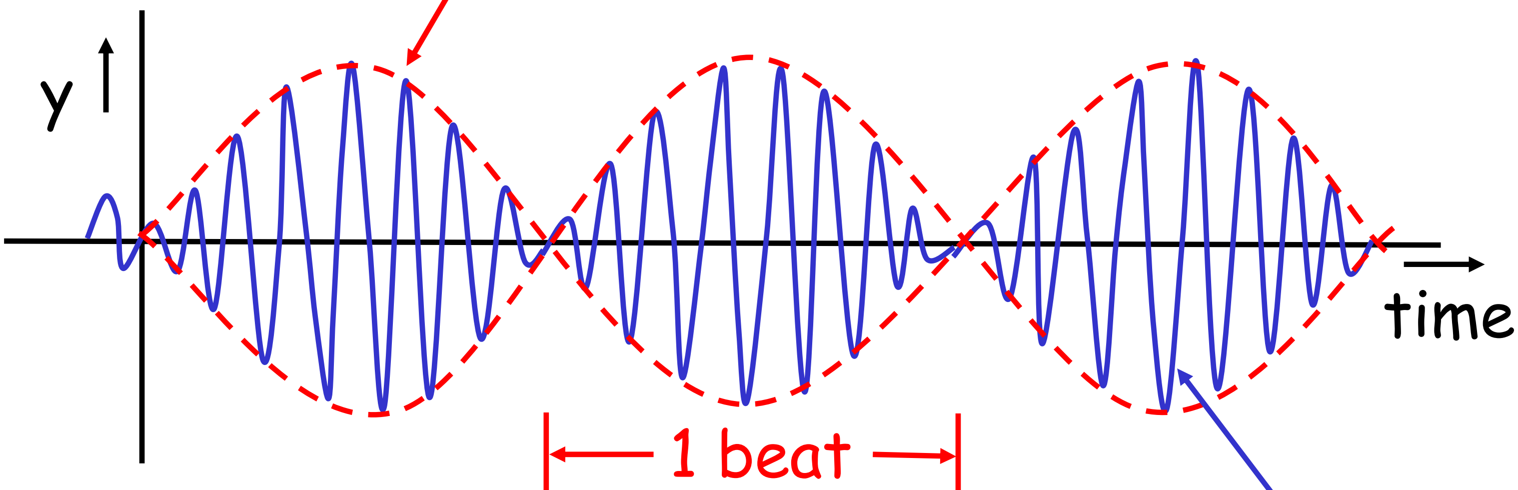
$$= \underbrace{2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right)}_{\text{slowly-varying amplitude}} \underbrace{\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)}_{\text{SHM at average frequency}}$$

slowly-varying amplitude

SHM at <u>average</u> frequency

Beats

$$2A\cos\left(\frac{\omega_1 - \omega_2}{2}t\right) \rightarrow f_b = |f_1 - f_2|$$



$$\cos\left(\frac{\omega_1 + \omega_2}{2}t\right)$$

Quick Quiz

While someone is tuning their guitar, you hear beats. If there is 0.5 seconds between beats, what is the difference in pitch between the string the notes that are being tuned?

a) 0.5 Hz
b) 1 Hz
c) 2 Hz
d) 4Hz

Interference of Waves

We looked at the effects of combining waves of the same amplitude and frequency but opposite directions (→ *standing waves*)

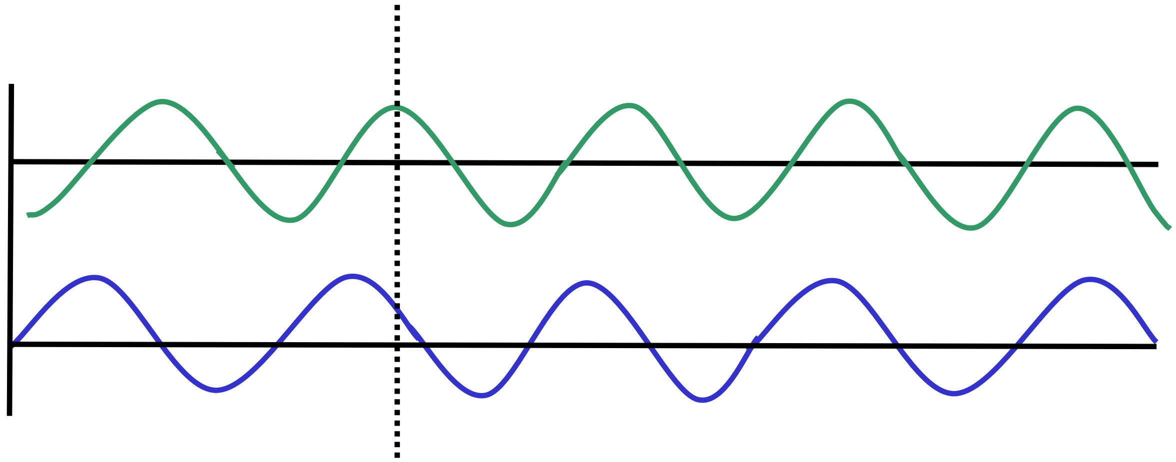
and 2 waves of the same amplitude but slightly different frequency (→ *beat frequencies*)

Now consider the effect of *phase differences* between two waves

Specifically: 2 waves, same amplitude, offset "phase" ϕ

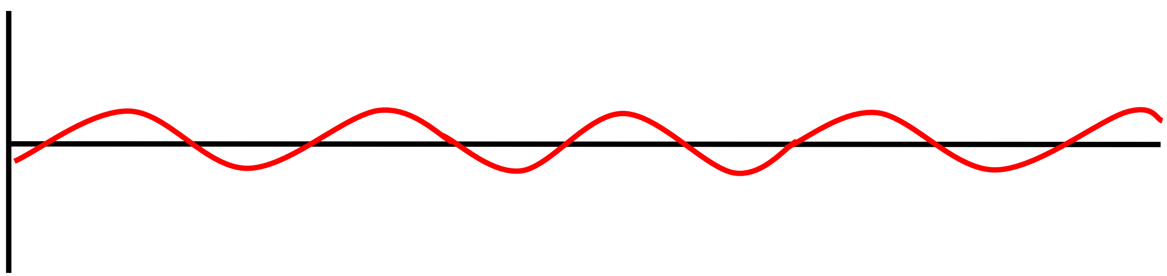
$$y_1 = A\sin(\omega t)$$

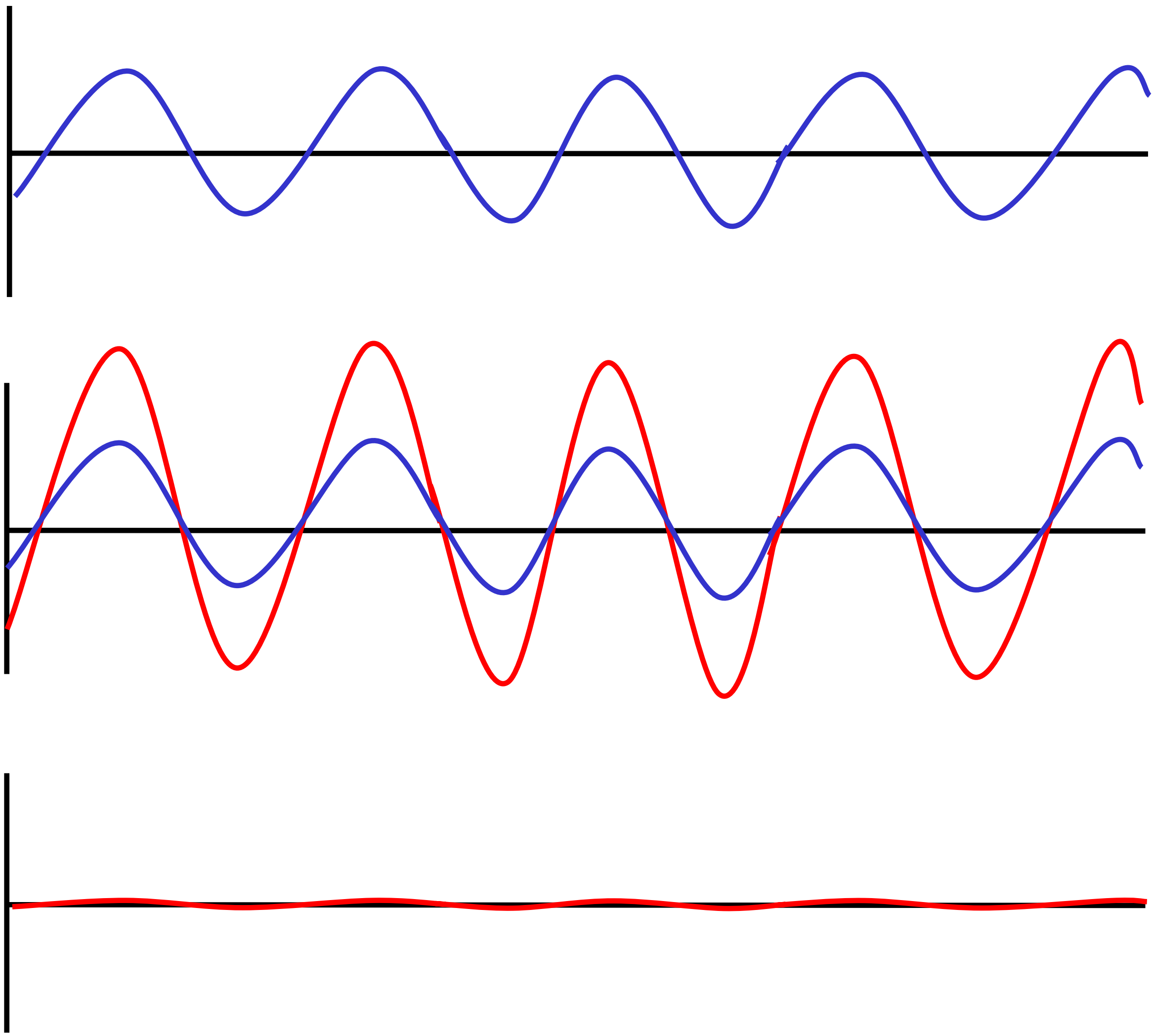
$$y_2 = A\sin(\omega t + \phi)$$



→ *Partial* destructive interference …

$$(y_1 + y_2)$$



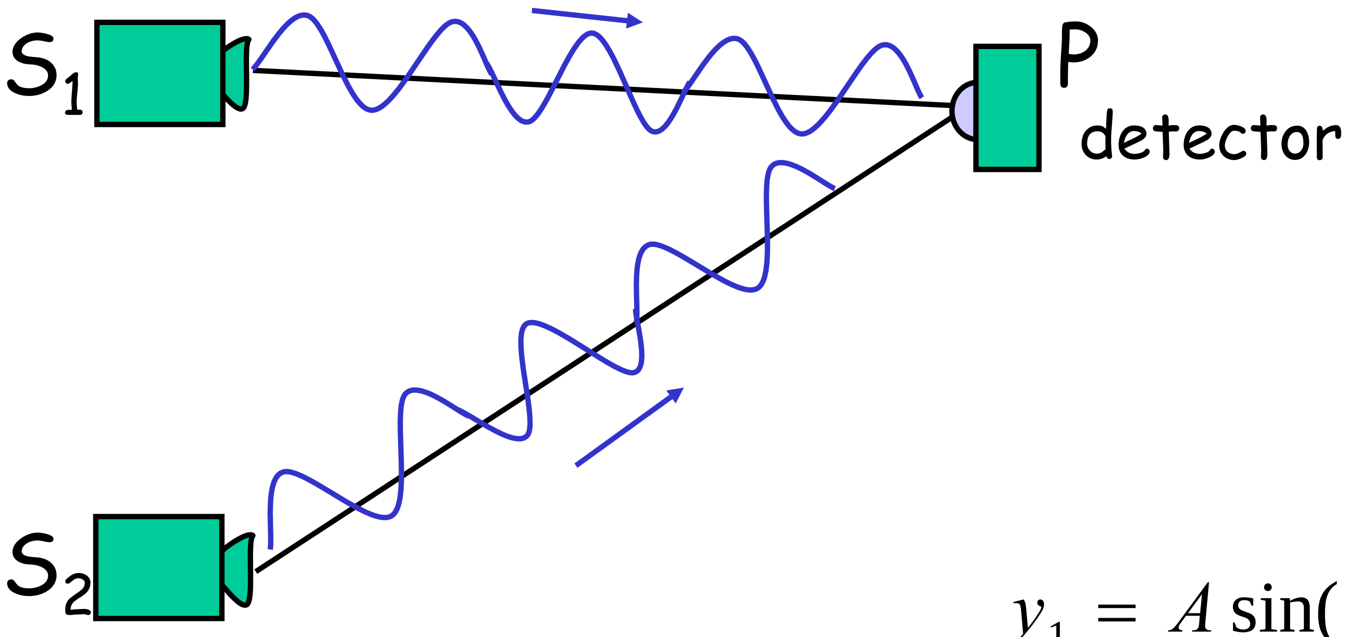


one original wave

If $\phi = 0$

If $\phi = \pi$ radians

Two sources emit waves <u>in phase</u> - but 2 paths have different lengths, so waves are offset when they arrive at detector



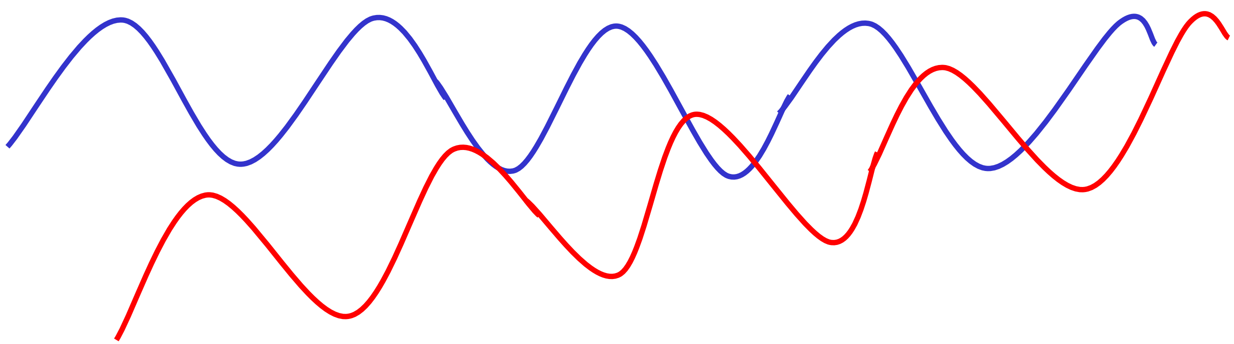
$$y_1 = A\sin(kr_1 + \omega t)$$

$$y_2 = A\sin(kr_2 + \omega t)$$

Phase difference $(kr_1 + \omega t) - (kr_2 + \omega t) = k(r_1 - r_2)$

so $\phi \equiv k\Delta r = \dfrac{2\pi}{\lambda}\Delta \mathrm{r}$ radians

The two waves are offset by $(\Delta r/\lambda)$ = fraction of one cycle



2 waves, of the *same frequency*, arrive out of phase.

Eg. $y_1 = A\sin(\omega t)$

$y_2 = A\sin(\omega t + \phi)$

Trigonometry:

sin a + sin b = 2 cos [(a-b)/2] sin [(a+b)/2]

Result: $y = y_1 + y_2$

$$\Rightarrow \quad y(x,t) = 2A\cos\left(\frac{\phi}{2}\right)\sin\left(\omega t + \frac{\phi}{2}\right)$$

Energy and intensity in the combined wave

The wave carries an **Intensity**

I = Power per unit area (Watts / m^2)

$$I \propto power \propto (amplitude)^2$$

$$y(total) = \underbrace{2A\cos(\varphi/2)}\cdot\sin(\omega t + \frac{\varphi}{2})$$

$$I = 4I_o \cos^2(\frac{\varphi}{2})$$

Where I_0 = intensity of each original wave

$\Phi = 0$ ➔ waves in phase

$\Phi = \pi$ ➔ waves out of phase

Under what conditions do we get a "maximum" or "minimum" intensity?

recall $\varphi = \dfrac{2\pi}{\lambda}\Delta r \quad \Rightarrow \quad \cos\dfrac{\varphi}{2} = \cos\left(\dfrac{\pi \cdot \Delta r}{\lambda}\right)$

maximum intensity

$$\cos\left(\frac{\varphi}{2}\right) = \pm 1$$

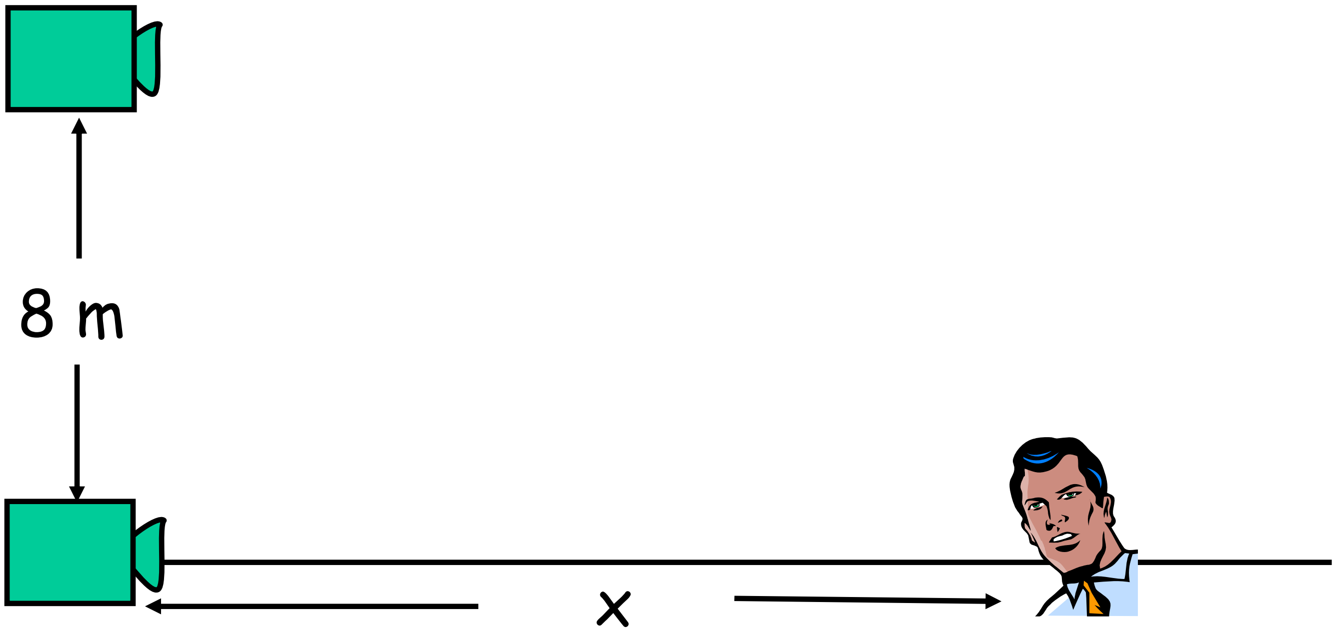
$$\frac{\varphi}{2} = n\pi \quad \text{and} \quad \varphi = \frac{2\pi}{\lambda}\Delta r$$

$$\rightarrow \quad \Delta r = n\lambda$$

minimum intensity

$$\cos\left(\frac{\varphi}{2}\right) = 0 \Rightarrow \Delta r = \left(n + \frac{1}{2}\right)\lambda$$

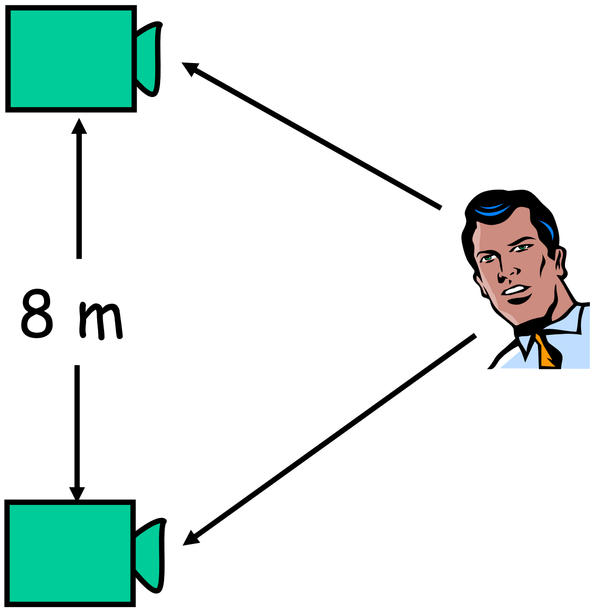
Example



2 speakers, in phase; f = 170 Hz

For what values of x is the sound intensity a maximum?

Quick Quiz

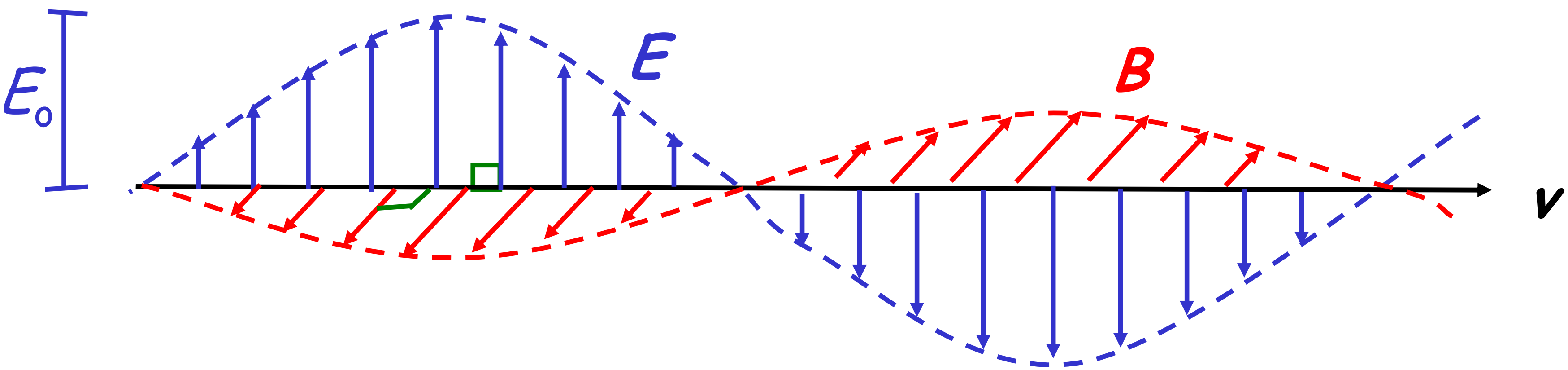


2 speakers, in phase; f = 170 Hz

The listener is exactly 3 metres from both speakers. If both speakers are wired such that they are exactly out of phase with each other, the listener will be:

a) at a node.
b) at an antinode.
c) somewhere between a node and an antinode.
d) Not enough information

Electromagnetic Waves



Electric field $$\vec{E}(x,t) = \vec{E}_o \sin(kx - \omega t)$$

$\vec{E}, \vec{B}, \vec{v}$ all perpendicular to each other

Electromagnetic Spectrum

λ (m)		f (Hz)
300	Radio	10^{6}
3	TV	10^{8}
3×10^{-3}	Microwave	10^{11}
3×10^{-6}	Infrared	10^{14}
7×10^{-7}	Visible	5×10^{14}
4×10^{-7}	Ultraviolet	
3×10^{-9}	X-Rays	10^{17}
3×10^{-12}	γ-Rays	10^{20}

Infrared

750 nm

Red

Yellow 600 nm

Green 550 nm

Blue 450 nm

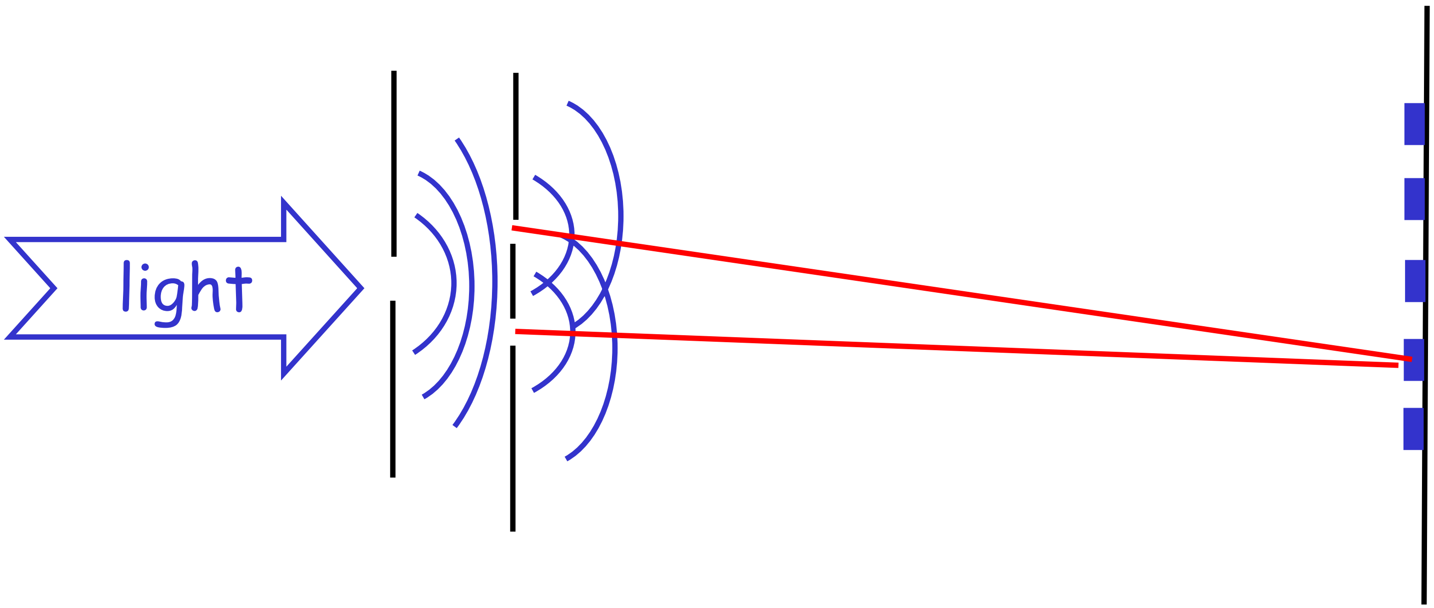
Violet

380 nm

Ultraviolet

Visible (= optical) part of the spectrum spans only a factor of ~2 in wavelength

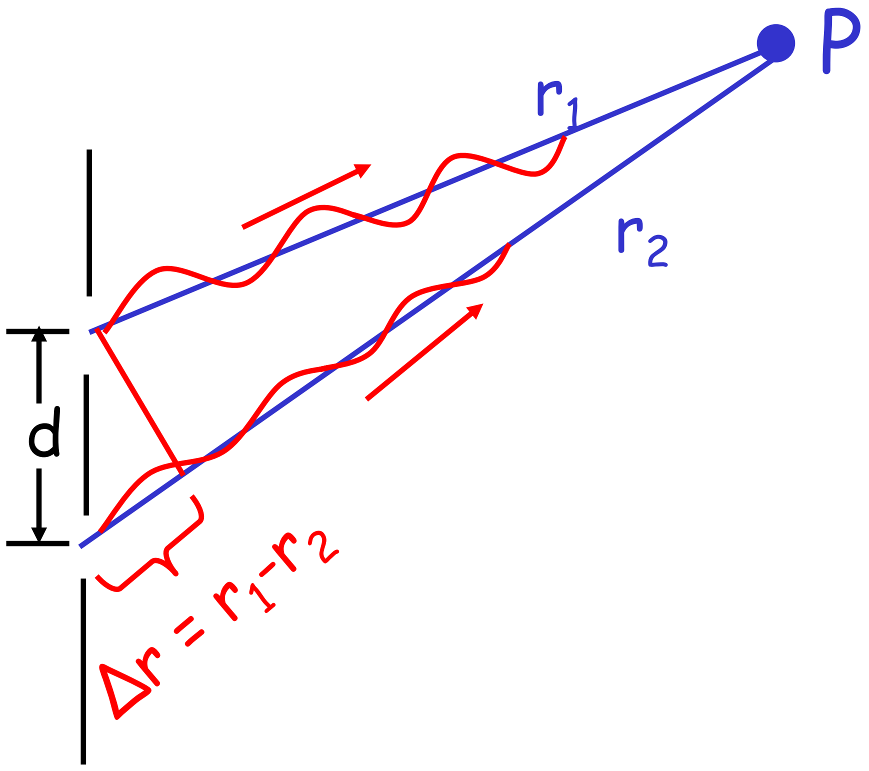
Experiment by Thomas Young (1801)

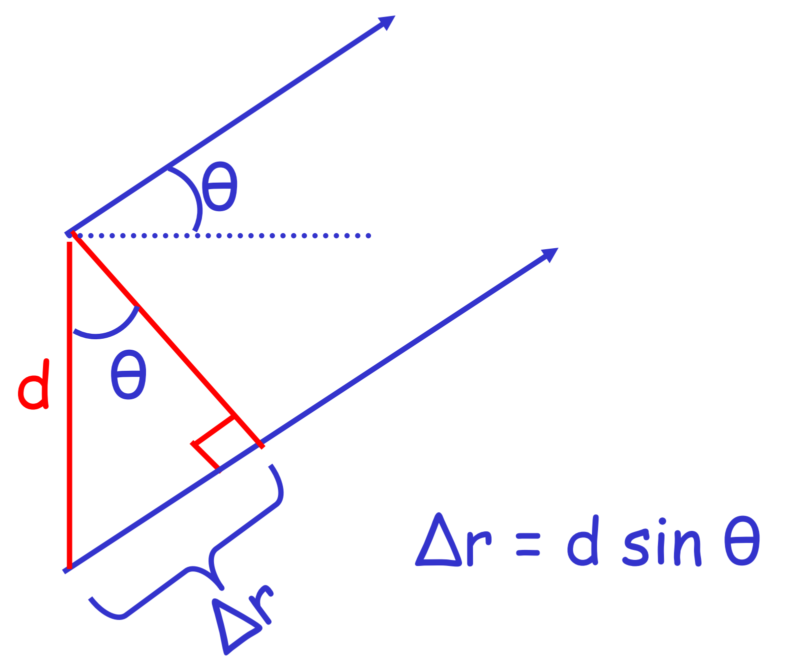


Result: Many bright "fringes" on screen, at specific angles; dark areas in between. *Classic demonstration that light had a "wave" nature.*

Slits act as wave sources that start out <u>in phase.</u>

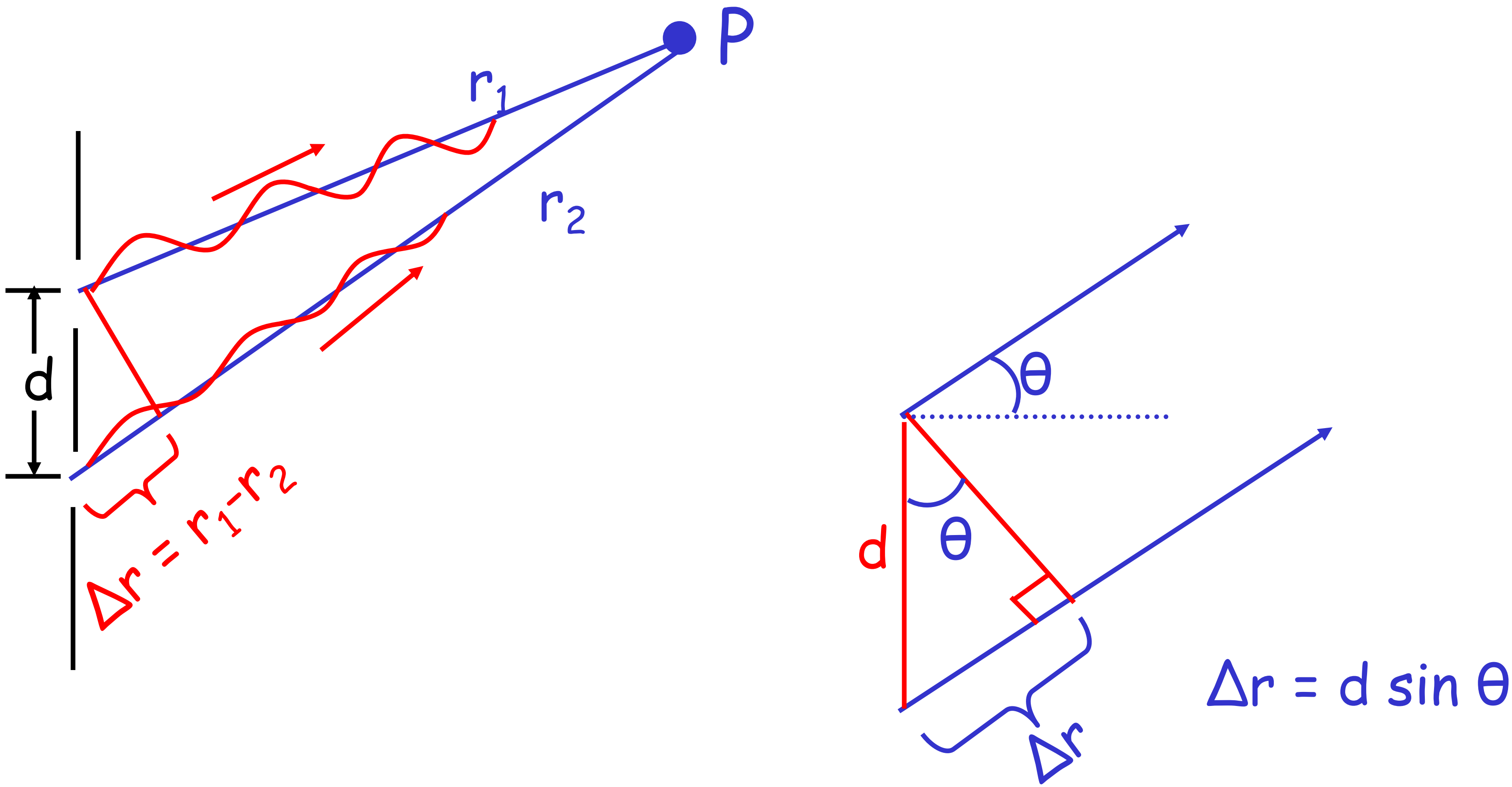
d << r and thus r_1, r_2 nearly parallel



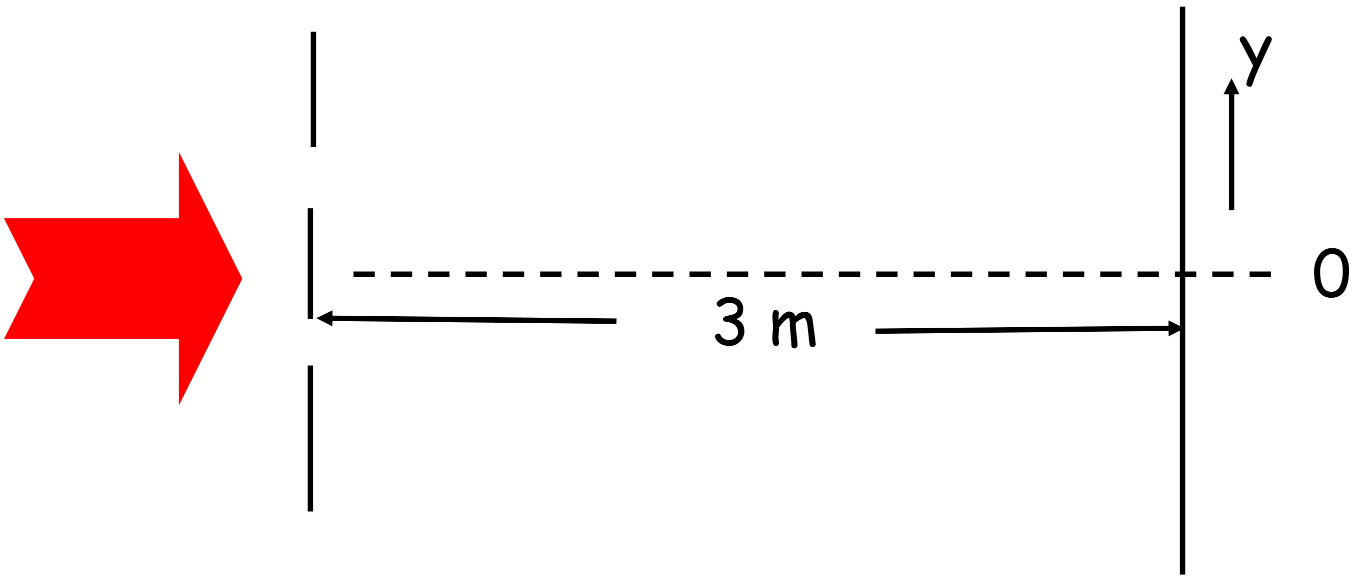


Constructive interference if $d \sin\theta = n\lambda$

Destructive interference if $d \sin\theta = (n+1/2)\lambda$



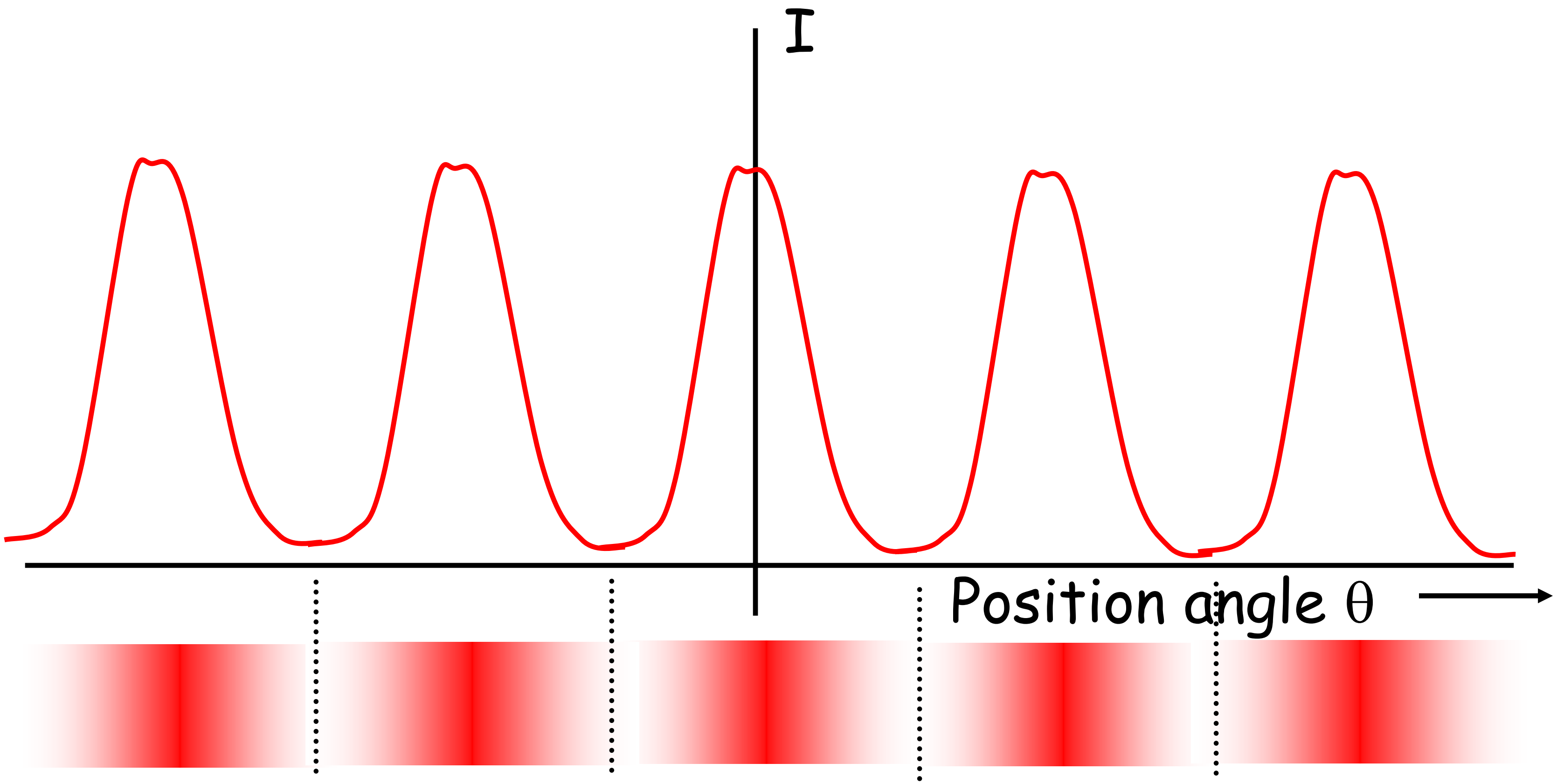
Example



2 slits, 0.20 mm apart. Red light (λ = 667 nm)

At what angles θ are

a) the bright fringes b) the dark lines ?



Note the fringes are equally spaced (in sin θ)