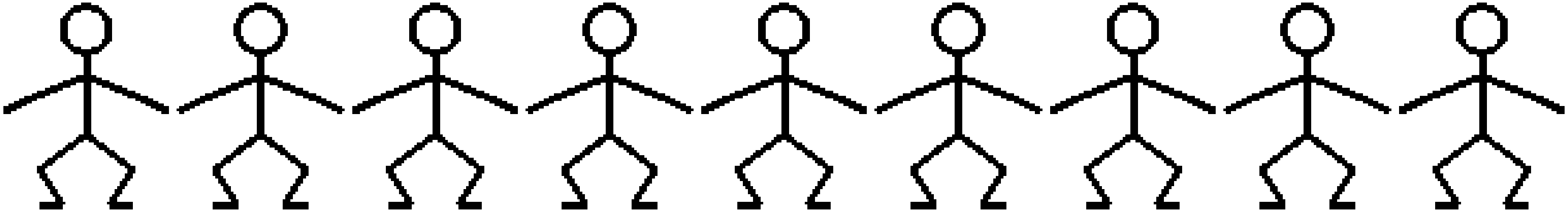
Wave Motion

- Qualitative properties of wave motion
- Mathematical description of waves in 1-D

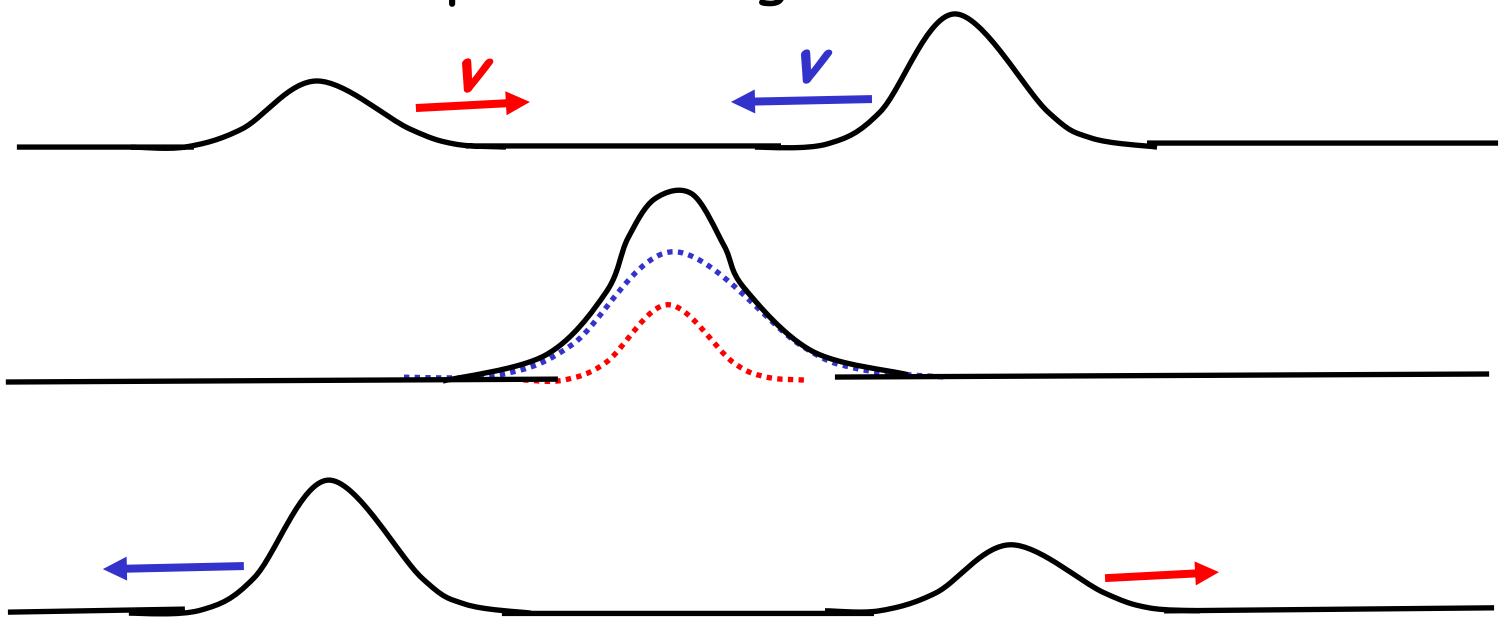


Principle of Superposition

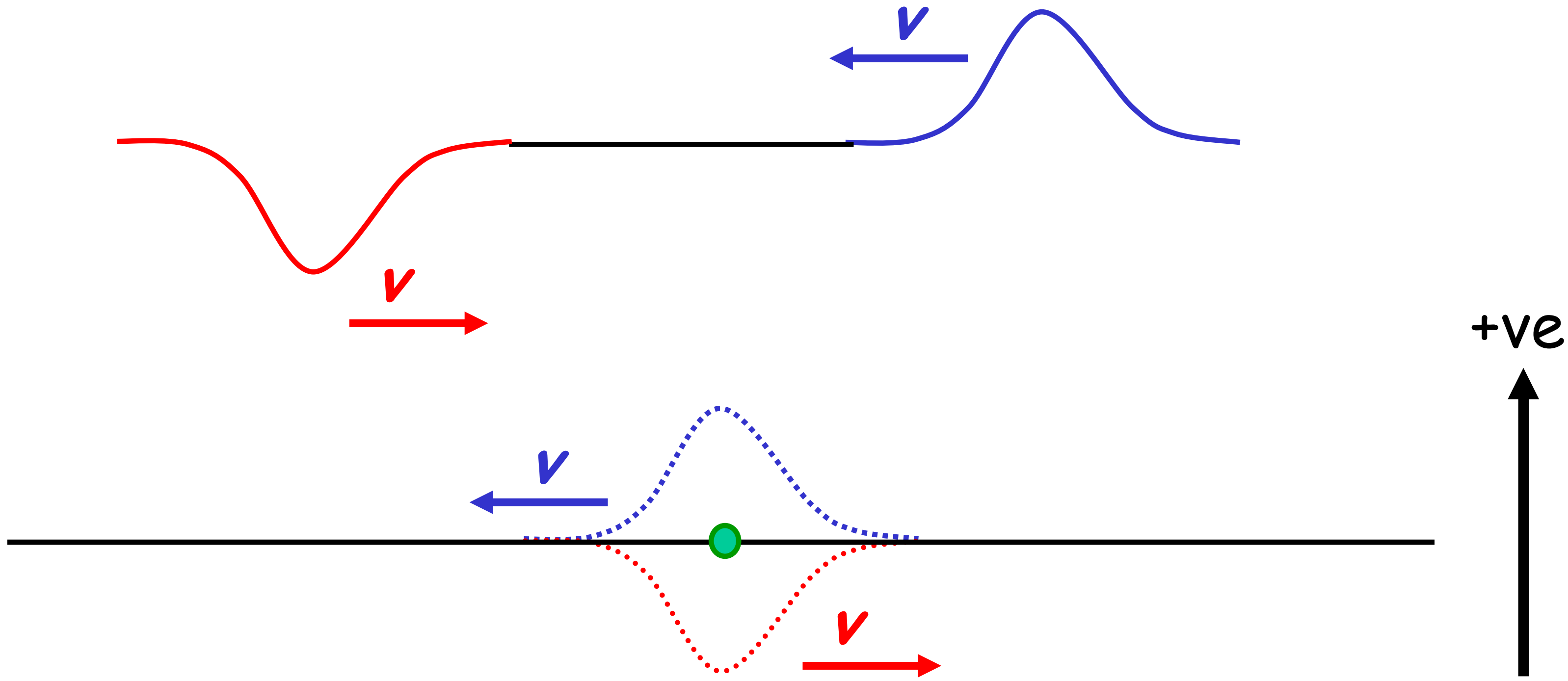
When two waves meet, the displacements add:

$$y_{\text{observed}}\ (x,t) = y_1(x,t) + y_2(x,t)$$

So, waves can pass through each other:



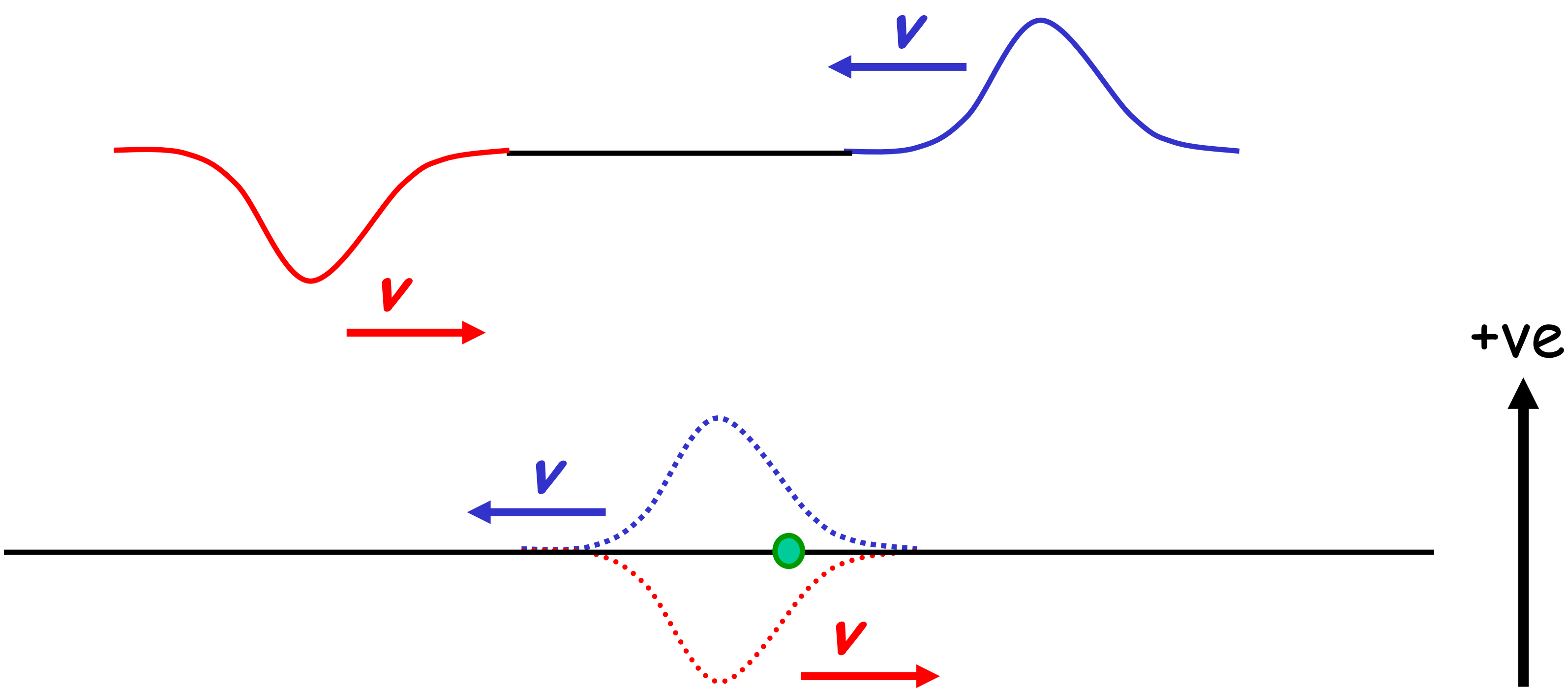
Quick Quiz



What is the velocity of the green particle on the string at the time when the string is flat?

A) Positive B) Negative C) Zero

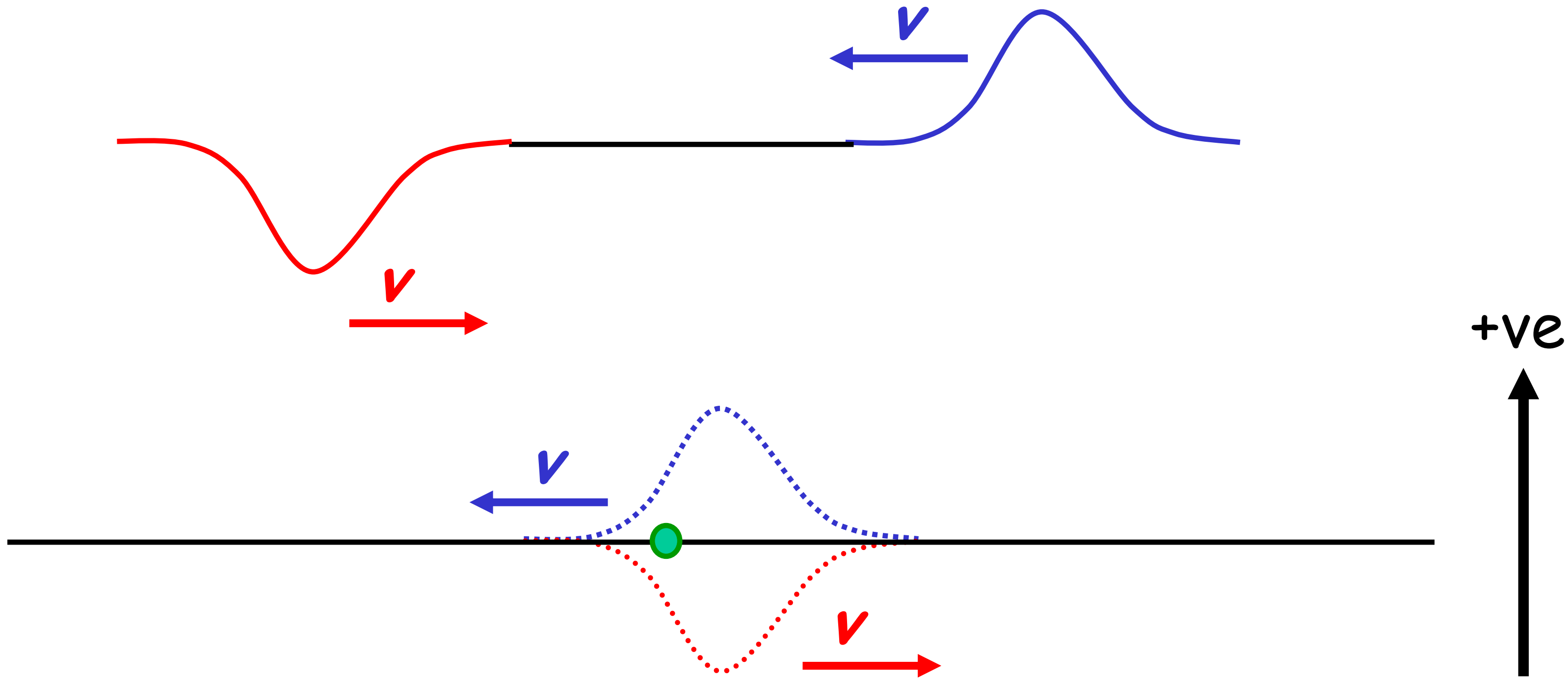
Quick Quiz



What is the velocity of the green particle on the string at the time when the string is flat?

A) Positive B) Negative C) Zero

Quick Quiz



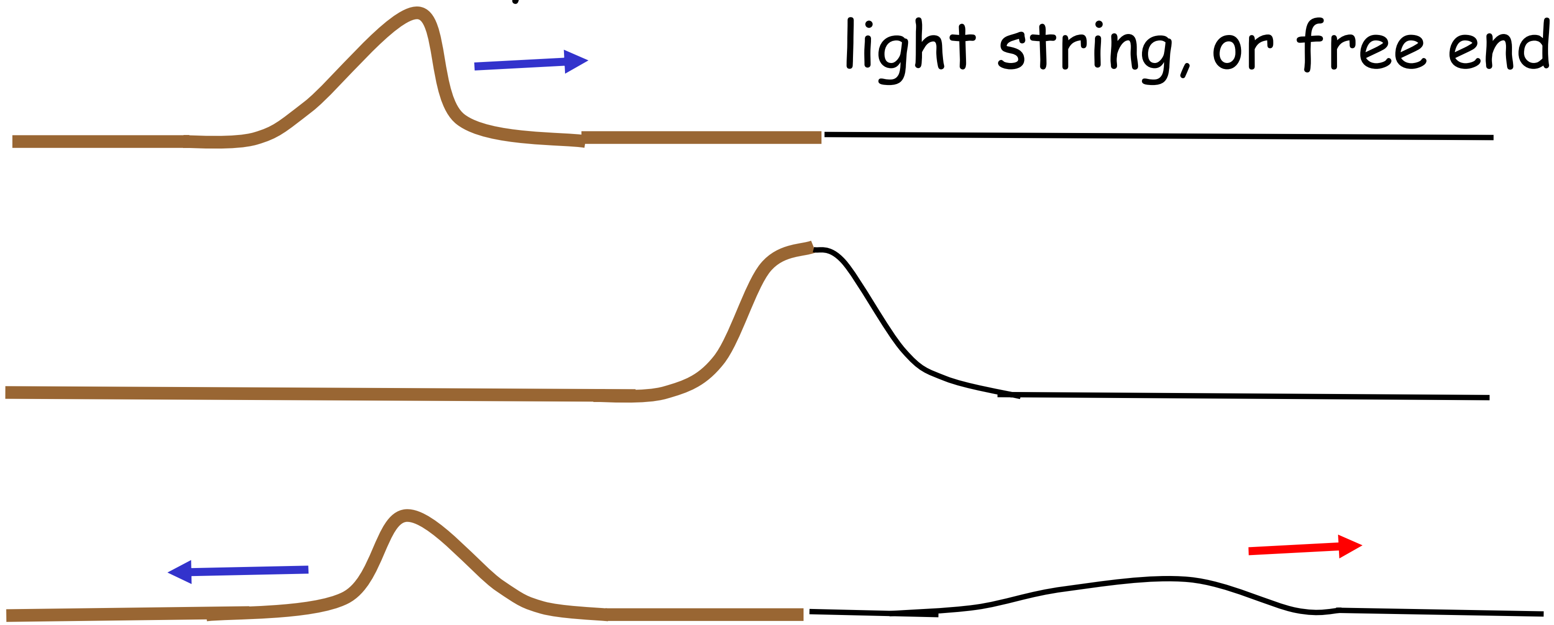
What is the velocity of the green particle on the string at the time when the string is flat?

A) Positive B) Negative C) Zero

Reflections

Waves (partially) reflect from any <u>boundary</u> in the medium:

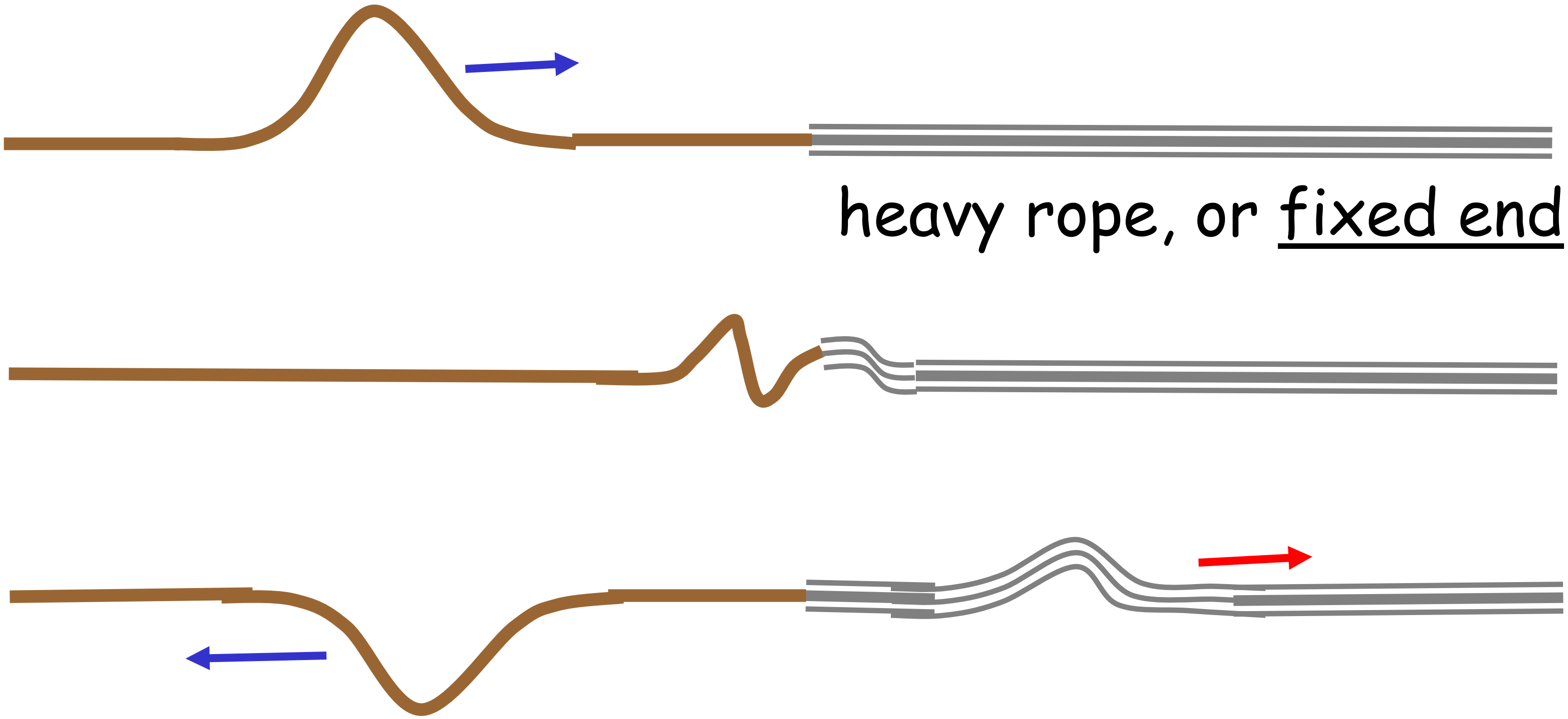
1) "Soft" boundary:



Reflection is <u>upright</u>

Reflections

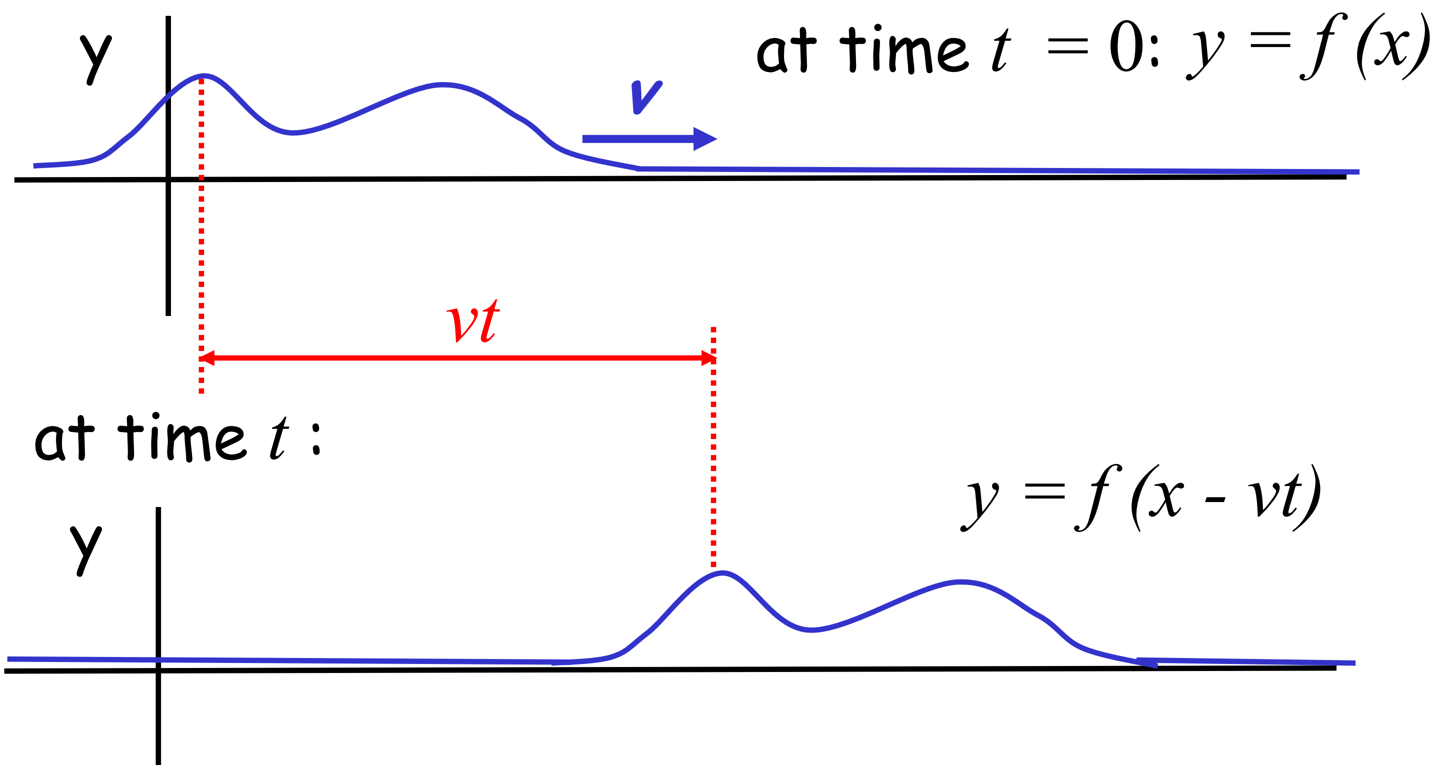
2) "Hard" boundary:



Reflection is inverted

Mathematical Characterisation of Traveling waves

Suppose the shape of the wave at t = 0, is given by some function y = f (x).



Note: y = y(x,t), a function of <u>two</u> variables;
f is a function of <u>one</u> variable

Non-dispersive waves:

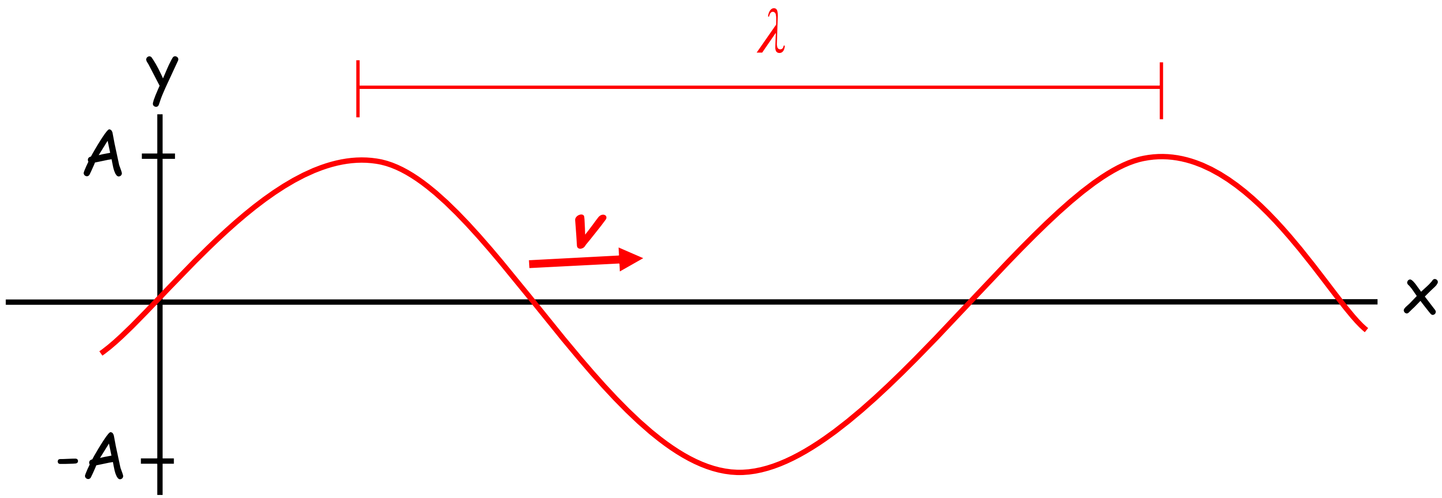
$$y\,(x,t) = f\,(x \pm vt)$$

+ sign: wave travels towards -x
- sign: wave travels towards +x

f(x) is any (smooth) function of <u>one</u> variable.

eg. $f(x) = A \sin(kx)$

<u>Sine waves</u>: (no longer a pulse)



Take $f(x) = A \sin(kx) = y(x, 0)$

Then $y(x,t) = f(x - vt)$

$= A \sin[k(x-vt)]$

Note λ = "wavelength": length (in metres) of one full wave

Also (kx) increases by 2π radians when x increases by λ

$$k = 2\pi / \lambda$$

Rewrite: $y = A \sin[kx - kvt]$

At fixed position x: $y = A \sin[\text{constant} - (kv)t]$

(Compare this to $y = A \cos(\omega t + \varphi)$)

→ Simple harmonic motion with angular frequency

$$\omega = kv$$

Thus, $y = A \sin(kx \pm \omega t)$

or $$y = A\sin(kx \pm \omega t - \varphi)$$ in general

A: amplitude

$k = 2\pi/\lambda$: "angular wavenumber" (radians/metre)

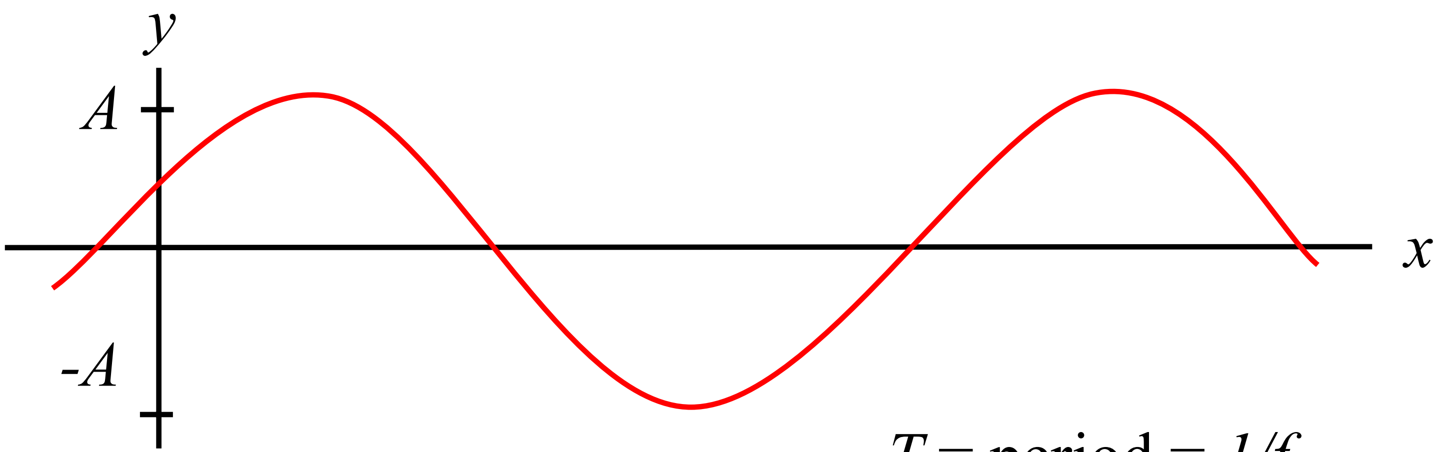
$\omega = 2\pi f$: "angular frequency" (radians/second)

φ : "phase constant" (radians)

Note: Wave speed, v = 1 wavelength / 1 period

$$v = f\lambda = \omega / k$$

Not-So-Quick Quiz



T = period = $1/f$

The wave shown is a graph of:

A) $y = A\,\sin(kx - \omega t)$ at time $t = T/6$
B) $y = A\,\sin(\omega t - kx)$ at time $t = 2T/3$
C) $y = A\,\sin(kx + \omega t - \pi/3)$ at time $t = 0$
D) all of the above
E) none of the above

Wave Velocity

The wave velocity is determined by the properties of the medium:

Transverse waves on a string:

$$v_{\text{wave}} = \sqrt{\frac{\text{tension}}{\text{mass/length}}} = \sqrt{\frac{F_T}{\mu}}$$

(proof from Newton's second law)

Electromagnetic wave (light, radio, etc.)

$$v_{\text{wave}} = \sqrt{\frac{1}{\mu_o \varepsilon_o}} = c$$

(proof from Maxwell's Equations for E-M fields)

Particle Velocities

particle displacement, y (x,t)

particle velocity, v_y = dy/dt (x held constant)

(Note that v_y is not the wave speed v !)

Acceleration, $a_y = \dfrac{dv_y}{dt} = \dfrac{d^2y}{dt^2}$

"Standard" sine wave:

$$y = A\sin(kx \mp \omega t - \varphi)$$

$$v_y = \frac{dy}{dt} = \mp\omega A\cos(kx \mp \omega t - \varphi)$$

$$a_y = \frac{dv_y}{dt} = -\omega^2 A\sin(kx \mp \omega t - \varphi)$$

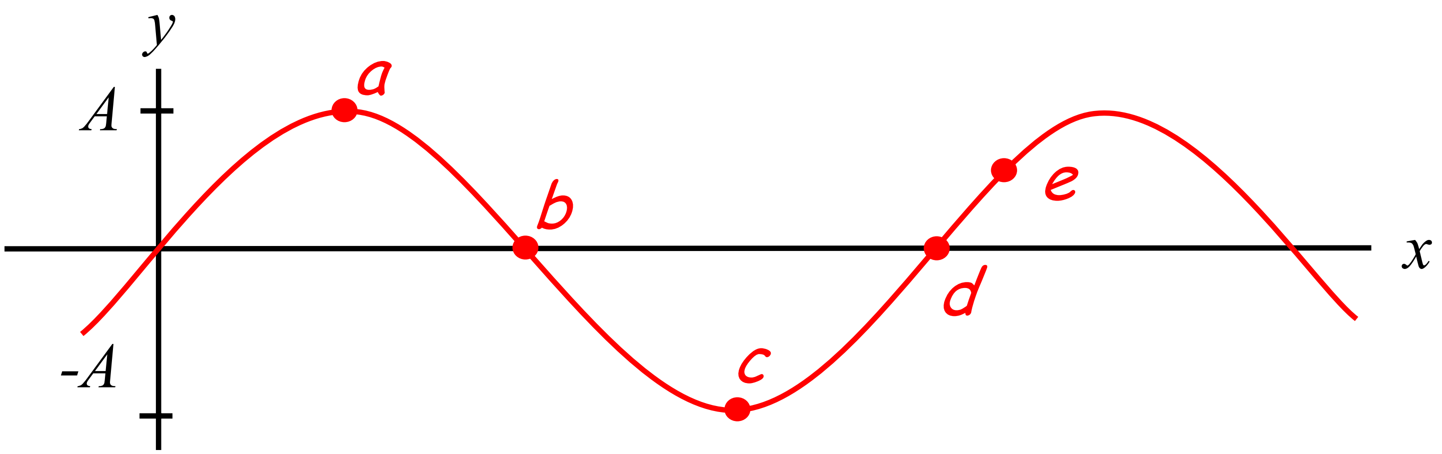
$$= -\omega^2 y$$

maximum displacement, $y_{\max} = A$

maximum velocity, $v_{\max} = \omega A$

maximum acceleration, $a_{\max} = \omega^2 A$

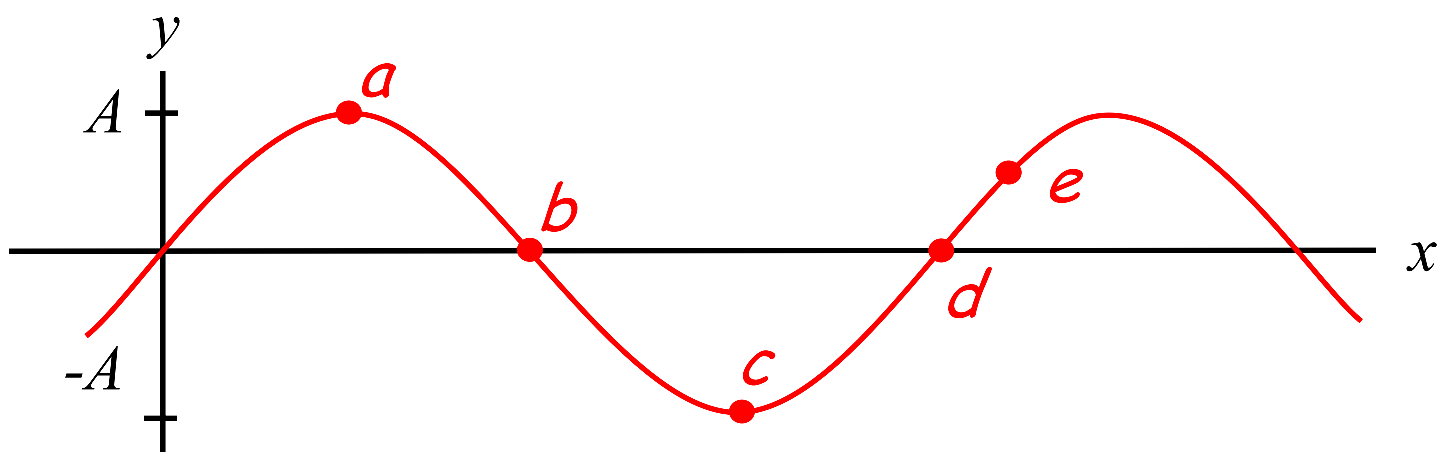
Example



Shown is a picture of a wave, $y = A\sin(kx - \omega t)$, at time $t = 0$.

i) Which particle moves according to $y = A\cos(\omega t)$?

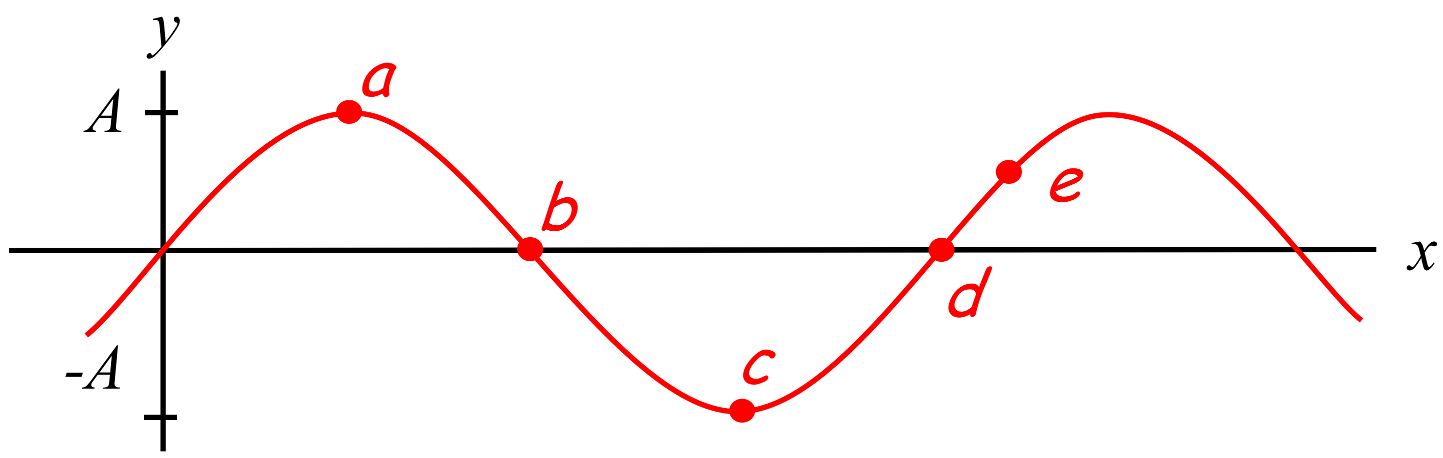
Example



Shown is a picture of a wave, $y=A\sin(kx-\omega t)$, at time $t=0$.

ii) If $y_e(t)=A\cos(\omega t+\phi_e)$ for particle e, what is ϕ_e ?

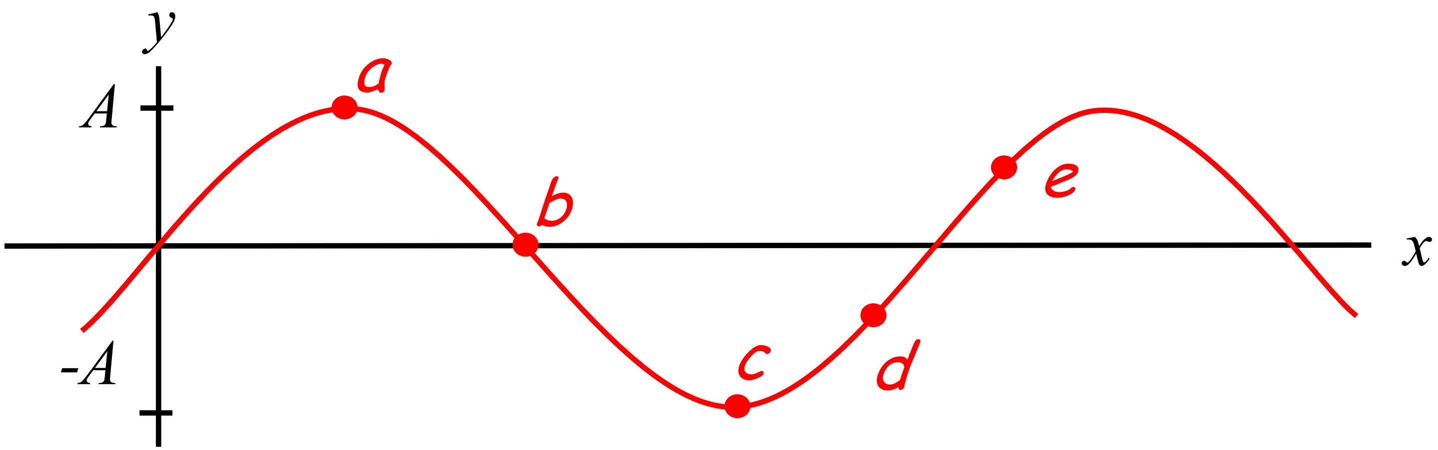
Quick Quiz



Shown is a picture of a wave, $y = A\sin(kx + \omega t)$, at time $t = 0$.

Which particle moves according to $y = A\sin(\omega t)$?

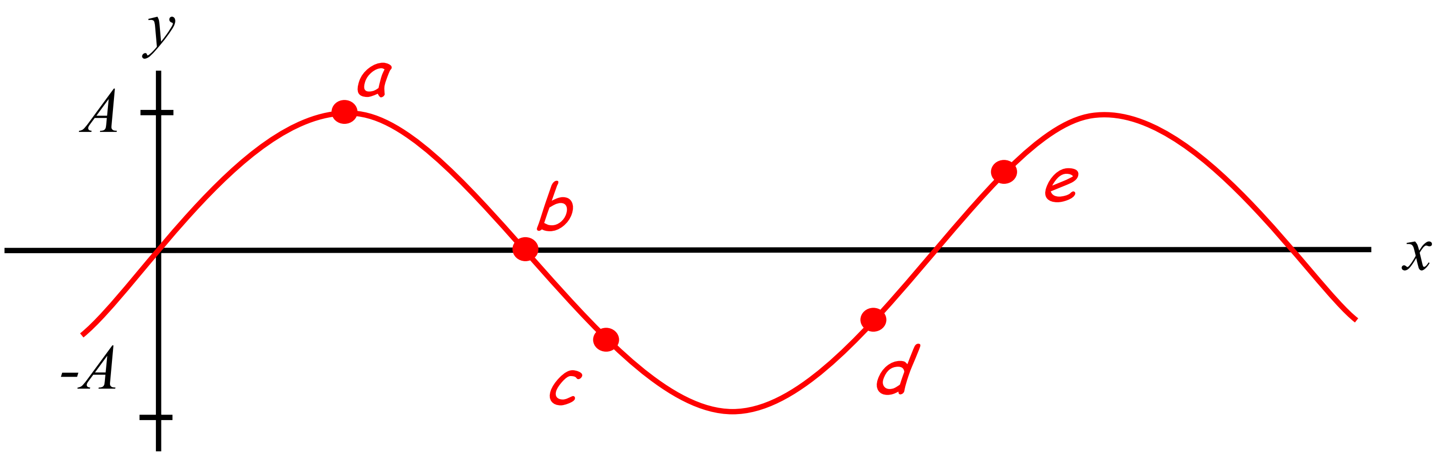
Quick Quiz



Shown is a picture of a wave, $y = A\sin(kx + \omega t)$, at time $t = 0$.

Which particle has the largest speed at this time?

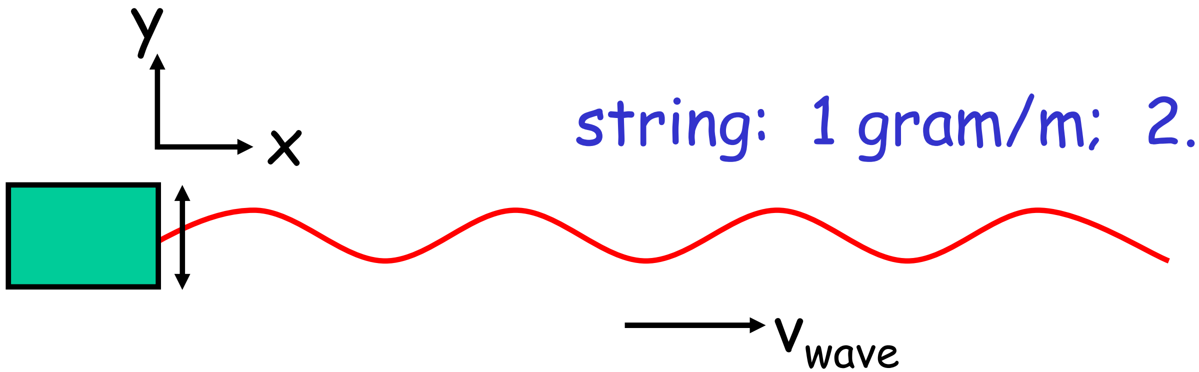
Quick Quiz



Shown is a picture of a wave, $y = A\sin(kx + \omega t)$, at time $t = 0$.

Which particle has the largest acceleration at this time?

Example



string: 1 gram/m; 2.5 N tension

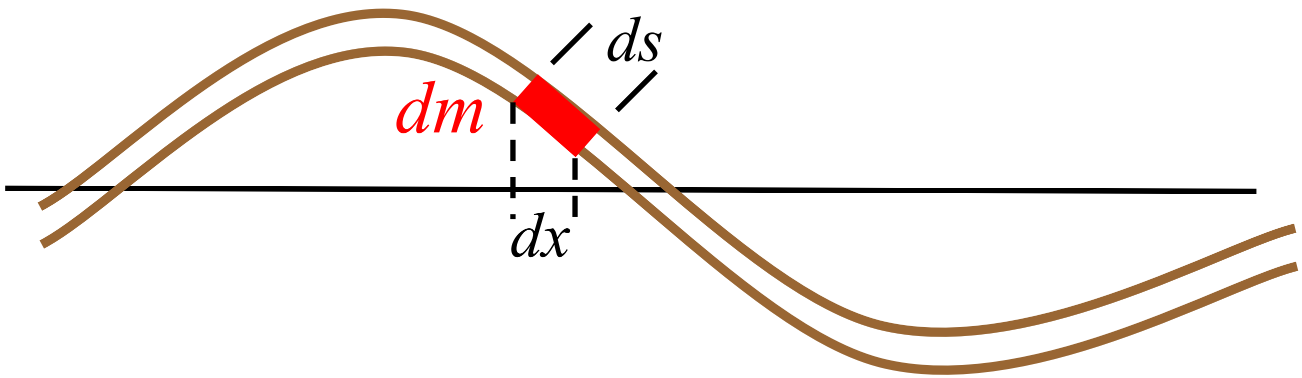
Oscillator:
50 Hz, amplitude 5 mm
$y\ (0, 0) = 0$.

Find: $y\ (x, t)$
$v_y\ (x, t)$ and maximum speed
$a_y\ (x, t)$ and maximum acceleration

Energy, Power

Energy, Power, etc. $\propto$ (amplitude)^2

Stretched rope, energy/unit length:



- Ignore difference between "ds", "dx" (small A, large λ)
- $dm = \mu\, dx$ (μ = mass/unit length)

The mass dm vibrates in simple harmonic motion. Its maximum kinetic energy is

$$dK_{max} = \tfrac{1}{2}(dm)(v_{max})^2$$
$$= \tfrac{1}{2}(dm)(\omega A)^2$$

The **average** kinetic energy is half this maximum value, but there is also an equal amount of potential energy in the wave. The total energy (kinetic plus potential) is therefore

$$dE = \tfrac{1}{2}(dm)\,\omega^2 A^2$$

To get the energy per unit length, replace the mass dm with the mass per unit length μ:

$$\frac{E}{(\text{unit length})} = \frac{1}{2}\mu\omega^2 A^2$$

__Power__: Energy travels at the wave speed *v*,

So
$$P = \left(\frac{\text{Energy}}{\text{length}}\right) \times v$$

waves on a string, $P = \frac{1}{2}\mu\omega^2 A^2 v$

Both the energy density and the power transmitted are proportional to the ***square*** *of the amplitude. This is a general property of sinusoidal waves.*