Simple Harmonic Motion

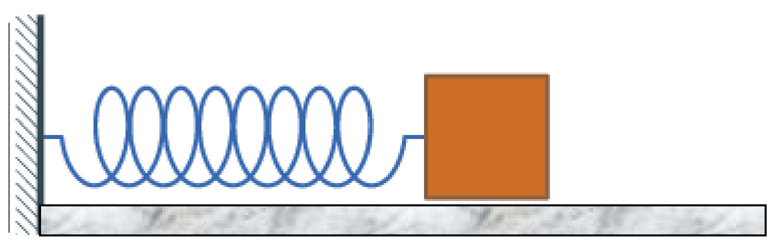
- Restating Hooke's law
- The equation of motion
- Phase, frequency, amplitude
- Simple Pendulum
- Damped and Forced oscillations
- Resonance

Harmonic Motion

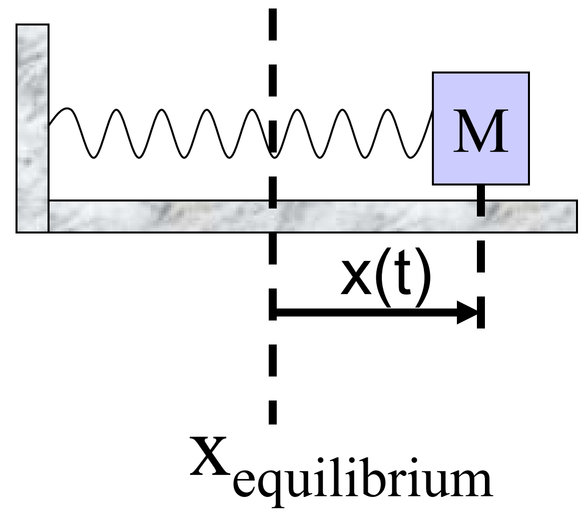
A lot of motion in the real world does not fit some of our earlier models (linear or circular motion, uniform acceleration).

Many phenomena are repetitive or oscillatory.

The most basic oscillation is the block/spring system:



x = distance from equilibrium.



Hooke's Law: $$F = -kx$$

$$\frac{F}{m} = -\frac{k}{m}x$$

$$a = -\frac{k}{m}x$$

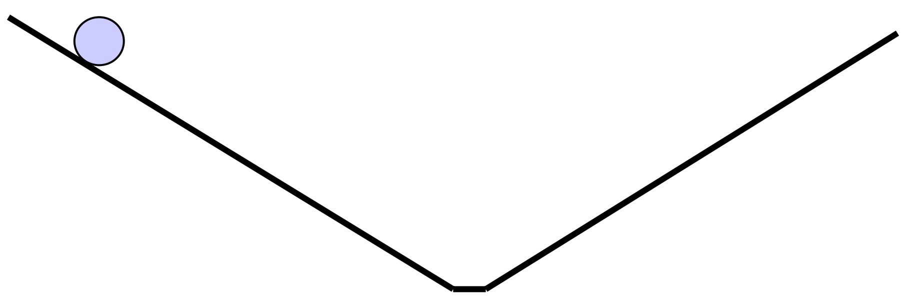
$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{k}{m}x$$

What is the *equation of motion*, x(t) ?

does $x(\mathrm{t}) = x_0 + v_0\mathrm{t} + \frac{1}{2} g\mathrm{t}^2$ work **?**

does $x(\mathrm{t}) = \mathrm{A}\,\mathbf{cos}\,(\omega \mathrm{t} + \phi)$ work **?**

- Is the motion of a bouncing ball described by SHM?



- Is this motion SHM?

Quick Quiz

Which of the following functions also describe simple harmonic motion?

A) $x(t) = e^{-t}$
B) $x(t) = \tan(\omega t)$
C) $x(t) = \sin(\omega t)$
D) All of the above
E) None of the above

Generic Simple Harmonic Oscillator

$$\frac{d^2x}{dt^2} = -\omega^2\, x(\mathrm{t})$$

$$\boxed{x(\mathrm{t}) = \mathrm{A}\,\mathbf{cos}(\omega\mathrm{t} + \varphi)}$$

➔ but what do *A*, ω, ϕ mean?

$$x = A\cos(\omega t + \varphi)$$

A = *amplitude* of the motion
= maximum value of x (x ranges from $+A$ to $-A$)

φ = *phase constant* -- gives the initial position at t=0

$$x(t=0) = x_o = A\cos\varphi$$

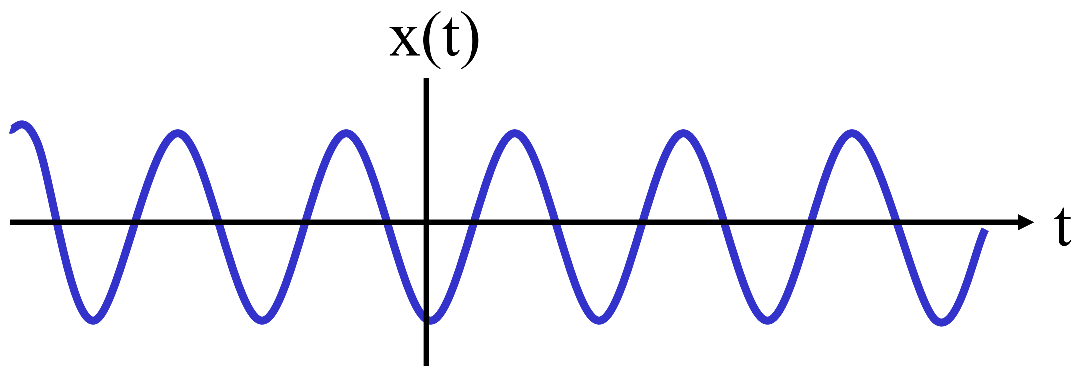
ω = *angular frequency* and is related to the *period* of the motion [radians/sec]

$$x = A\cos(\omega t + \varphi)$$

- This quantity in brackets is an *angle*, called the *phase angle* (or just the *phase*) of the motion

- x=A at phase angles of multiples of 2π radians:

  ... ,–4π, –2π, 0, 2π, 4π, …



$$x = A\cos(\omega t + \varphi)$$

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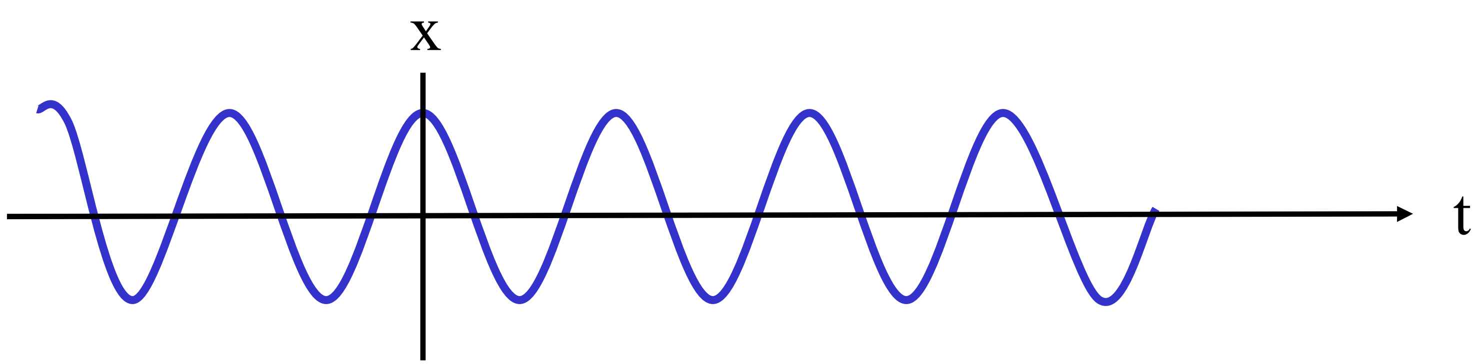


The period *T* is the time needed to repeat the cycle:

At $t = 0$, position is $x_0 = A \cos \phi$

One period later, at $t = T$, position is

$x = A \cos (\omega T + \phi)$

In time T, the phase angle $(\omega t+\phi)$ has increased by (2π) radians

$$\omega T = 2\pi \ \ radians \quad \Rightarrow \quad \omega = 2\pi / T$$

Two types of frequency!

Angular frequency: $\omega = 2\pi / T$ in [rad/s]

Frequency: $f = 1/T$ in [Hz] or [s^{-1}]

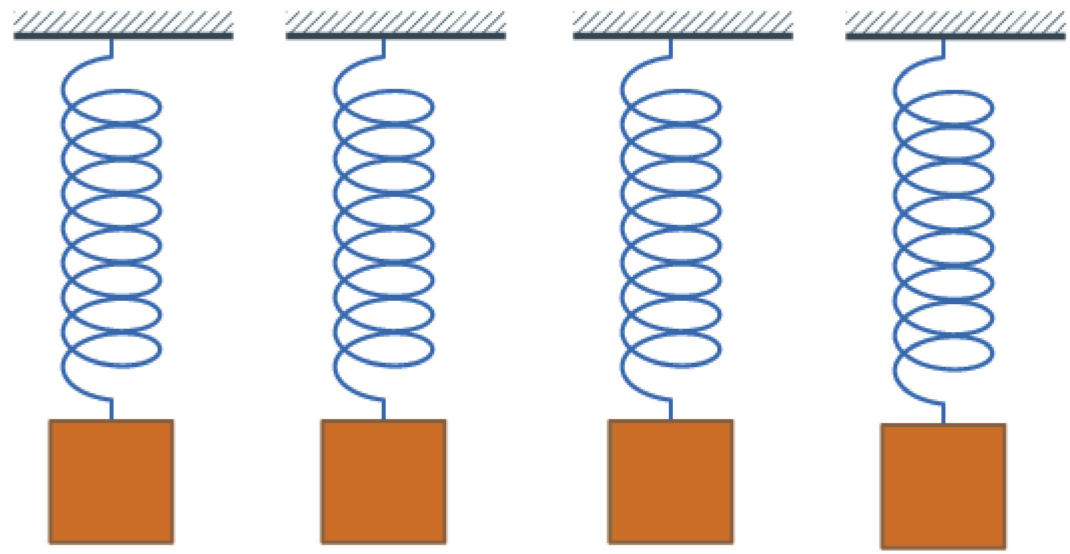
cycles per second

Quick Quiz

A mass hung from a spring is made to oscillate vertically. The mass goes from its highest position to its lowest position in 2 seconds. What is the angular frequency of this oscillation?

A) 1.5 rad/s
B) 3 rad/s
C) 1.5 Hz
D) 3 Hz

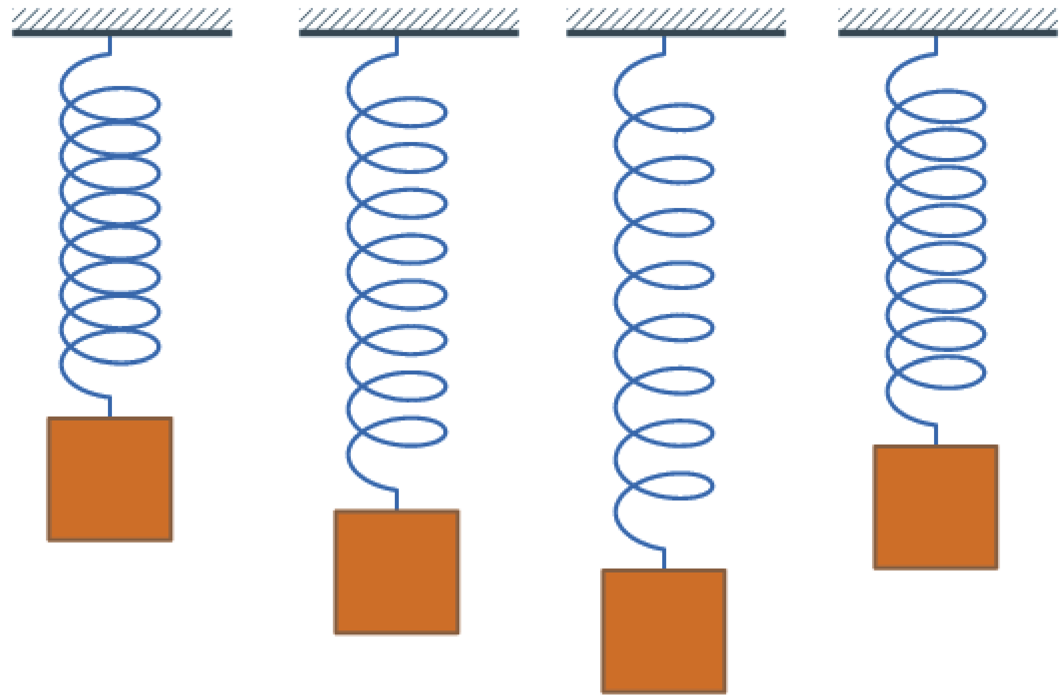
Quick Quiz



What is the difference between these oscillators?

A) Phase constant
B) Amplitude
C) Angular Frequency
D) More than one of the above
E) None of the above

Quick Quiz

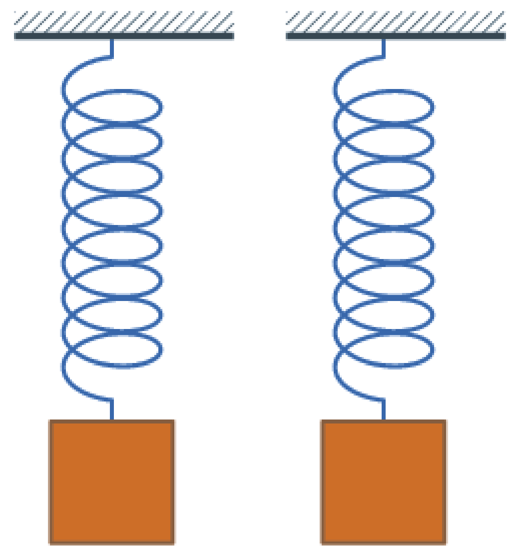


What is the difference between these oscillators?

A) Phase constant
B) Amplitude
C) Angular Frequency
D) More than one of the above
E) None of the above

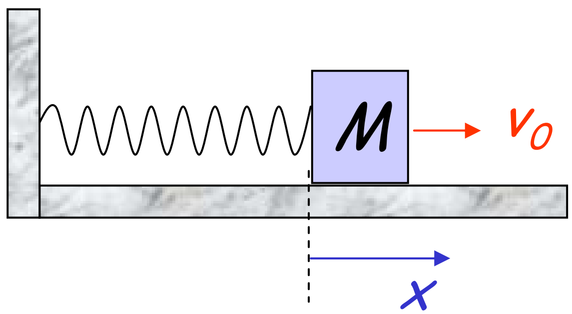
Example

What is the ratio of the periods for these two oscillators?



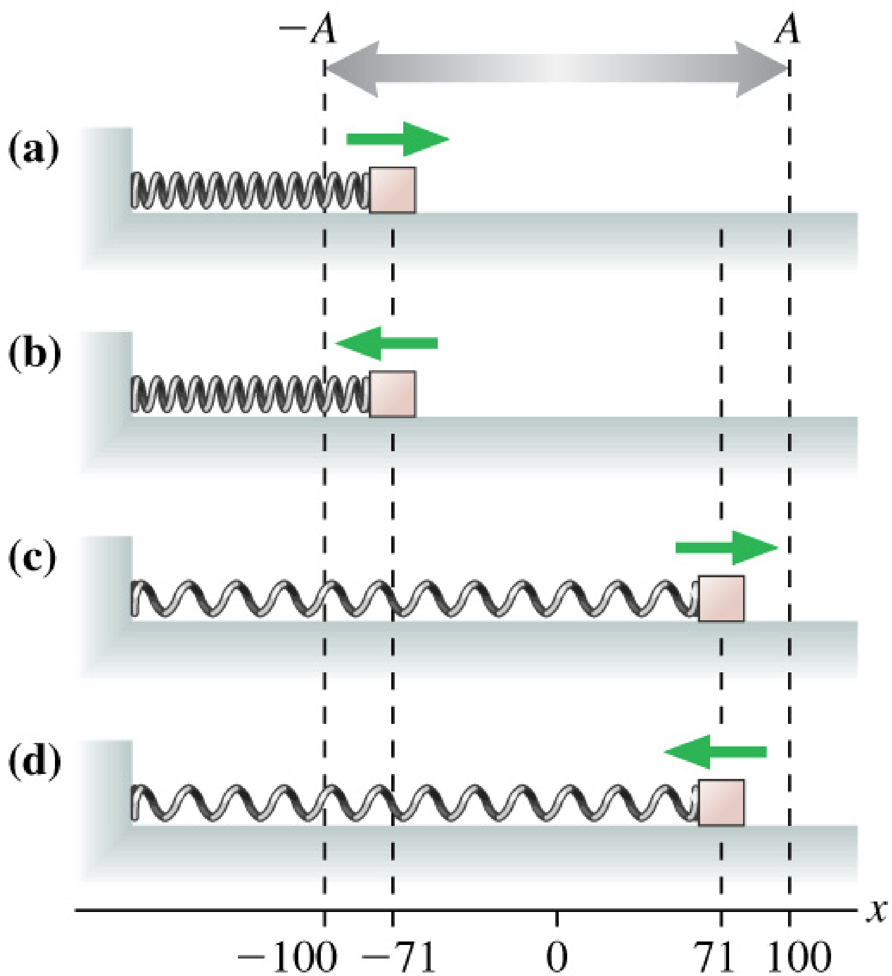
Example

The block is at its equilibrium position and is set in motion by hitting it (and giving it an initial velocity) at time $t = 0$. Its motion is SHM with amplitude 5 cm and period 2 seconds. Write the function $x(t)$.

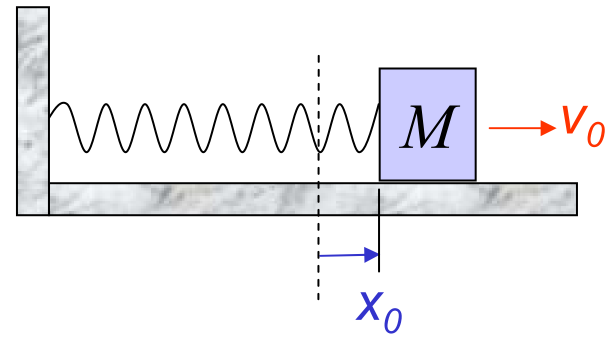


Quick Quiz

The figure shows four oscillators at $t = 0$. Which one has the phase constant $\varphi_0 = \pi/4$ rad?



Quick Quiz



The block is at x_0 = +5 cm, with positive velocity v_0, at time t = 0. Its motion is SHM with amplitude 10 cm and period 2 seconds. If x(t) = A cos (ωt + ϕ), the phase constant f should be:

A) 0°
B) 30°
C) 60°
D) -30°
E) -60°

Velocity and Acceleration

$$x(t) = A\cos(\omega t + \varphi)$$

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \varphi)$$

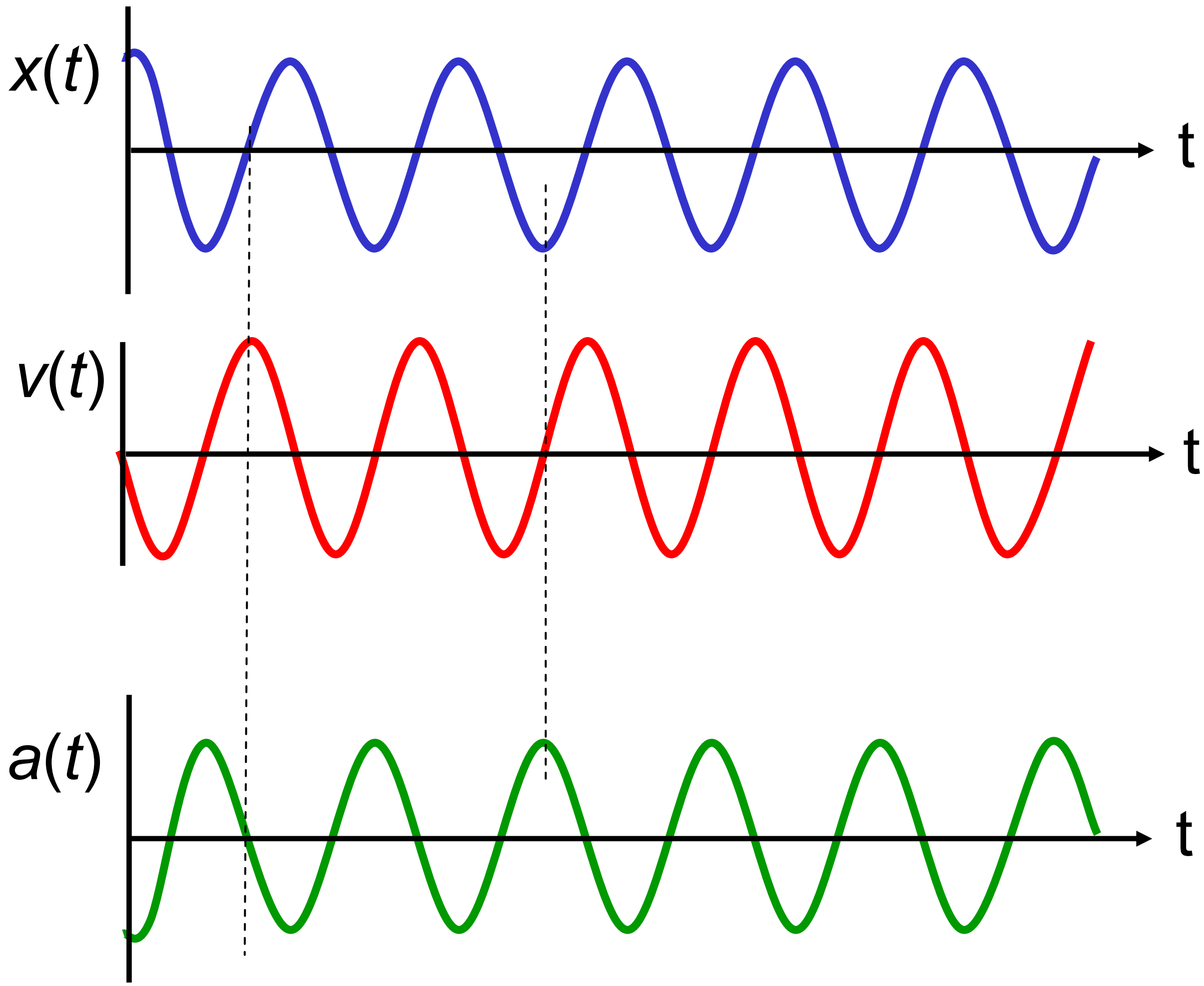
$$a(t) = \frac{dv}{dt} = -A\omega^2\cos(\omega t + \varphi) = -\omega^2 x$$

$$a(t) = -\omega^2 x(t)$$

Note : $v_{\max} = A\omega$

$a_{\max} = A\omega^2$

Position, Velocity and Acceleration



Question:

Where in the motion is the velocity largest?

Where in the motion is acceleration largest?

Quick Quiz

An object moves with simple harmonic motion. If the amplitude and the period are both doubled, the object's maximum speed is

A. quartered.
B. halved.
C. quadrupled.
D. doubled.
E. Unchanged.

Example

An object oscillates with SHM along the *x*-axis. Its displacement from the origin varies with time according to the equation:

$$x(t)=(4.0m)\cos(\pi t+\pi/4)$$

where *t* is in seconds and the angles in radians.

a) determine the amplitude
b) determine the frequency
c) determine the period
d) its position at t=0 sec
e) calculate the velocity at any time, and the v_{max}
f) calculate the acceleration at any time, and a_{max}

Back to Hooke's Law: $F = -kx$

$$\frac{d^2x}{dt^2} = -\omega^2 x \quad \text{where } \omega^2 = \frac{k}{m}$$

Generic SHO

Hooke's SHO

$$\omega = 2\pi / T \quad \rightarrow \quad T = 2\pi\sqrt{\frac{m}{k}}$$

Example

A 7.0 kg mass is hung from the bottom end of a vertical spring fastened to the ceiling. The mass is set into vertical oscillations with a period of 2.6 s.

Find the spring constant (aka force constant of the spring).

Example

A block with a mass of 200g is connected to a light spring with a spring constant k=5.0 N/m and is free to oscillate on a horizontal frictionless surface.

The block is displaced 5.0cm from equilibrium and released from rest.

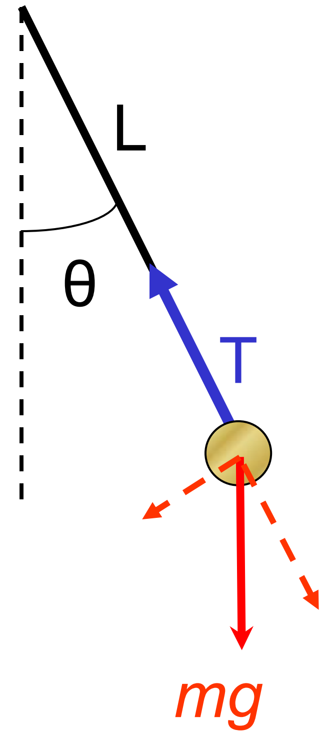
a) find the period of its motion
b) determine the maximum speed of the block
c) determine the maximum acceleration of the block

Simple Pendulum

Gravity is the "restoring force" taking the place of the "spring" in our block/spring system.

Instead of *x*, measure the displacement as the arc length *s* along the circular path.

Write down the *tangential component* of F=ma:

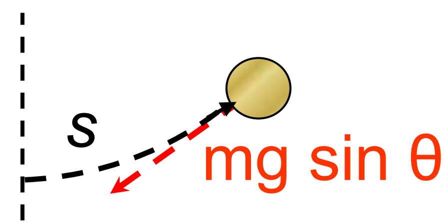


Restoring force $= mg \sin\theta$

$$-mg\sin\theta = ma_t = m\frac{d^2 s}{dt^2}$$

But $s = L\theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$



SHM: $$\frac{d^2x}{dt^2} = -\omega^2 x \propto -x$$

Simple pendulum: $$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

The pendulum is *not* a simple harmonic oscillator!

However, take *small* oscillations:

$\sin\theta \cong \theta$ (radians) if θ is small.

Then $$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta \approx -\frac{g}{L}\theta$$

For small θ: $$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

This looks like $\frac{d^2x}{dt^2} = -\omega^2 x$, with angle θ instead of *x*.

The pendulum oscillates in SHM with an *angular frequency*

$$\boxed{\omega = \sqrt{\frac{g}{L}}}$$

and the period is $2\pi/\omega$, *etc.*

Question: How high is the ceiling?

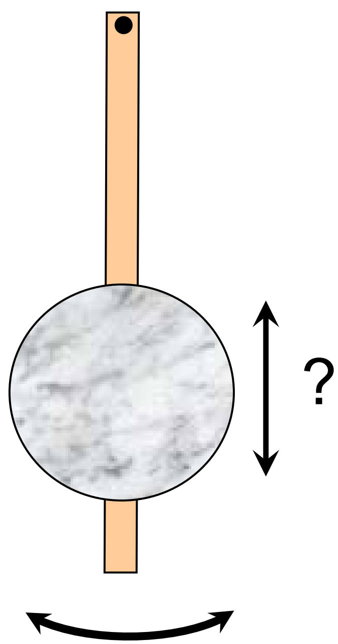
Quick Quiz

What happens to the period of a simple pendulum if the mass is doubled?

a) Increases by a factor of two
b) Increases by a factor of $\sqrt{2}$
c) Stays the same
d) Decreases by a factor of two
e) Decreases by a factor of $\sqrt{2}$

Quick Quiz

Pendulum clocks ("grandfather clocks") often have a swinging arm with an adjustable weight. Suppose the arm oscillates with T=1.05sec and you want to adjust it to 1.00sec. Which way do you move the weight?



Quick Quiz

A simple pendulum hangs from the ceiling of an elevator. If the elevator accelerates upwards, the period of the pendulum:

a) Gets shorter
b) Gets larger
c) Stays the same

Energy of a SHO

Recall for a spring:

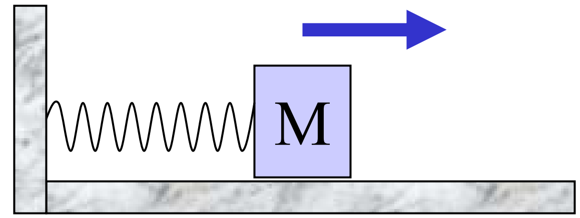
$$E_{Tot} = K + U$$
$$= 1/2mv^2 + 1/2kx^2$$

But we know that:

$$x = A \cos(\omega t+\varphi)$$
$$v = -A \omega \sin(\omega t+\varphi)$$

Energy in SHM

Look again at the block & spring



$$K = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m \omega^2 A^2 \sin^2(\omega t + \phi)$$

$$U = \tfrac{1}{2} k x^2 = \tfrac{1}{2} k A^2 \cos^2(\omega t + \phi)$$

$$= k!$$

$$K + U = \tfrac{1}{2} A^2 \left(m\omega^2 \sin^2(\omega t + \phi) + k \cos^2(\omega t + \phi) \right)$$

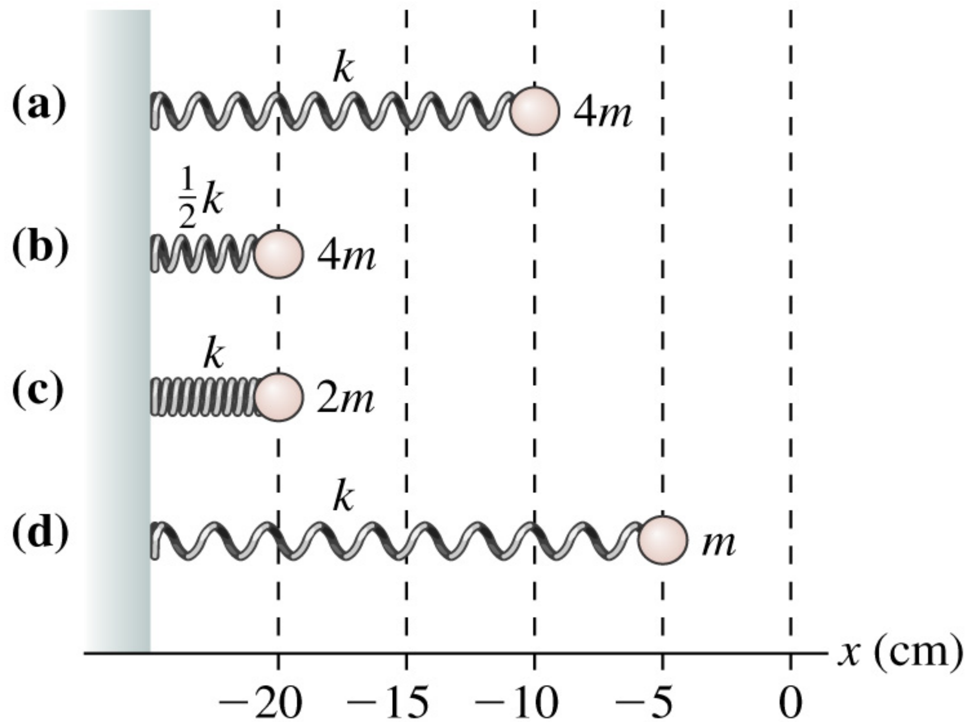
$$= \tfrac{1}{2} k A^2 \left(\sin^2(\omega t + \phi) + \cos^2(\omega t + \phi) \right)$$

$$= \tfrac{1}{2} k A^2 = \text{a constant (total mechanical energy)}$$

$$\Longrightarrow \quad E_{tot} = \frac{1}{2} k A^2 = \frac{1}{2} (m \omega^2) A^2 = \frac{1}{2} m v_{\max}^2$$

Quick Quiz

Four springs have been compressed from their equilibrium position at $x = 0$ cm. When released, they will start to oscillate. Rank in order, from highest to lowest, the maximum speeds of the oscillators.



A. c > b > a = d

B. c > b > a > d

C. d > a > b > c

D. a = d > b > c

E. b > c > a = d

Quick Quiz

Suppose you double the amplitude of the motion, what happens to the maximum speed ?

a) Doubles
b) Four times larger
c) Doesn't change

Quick Quiz

Suppose you double the amplitude of the motion, what happens to the maximum acceleration?

a) Doubles
b) Four times larger
c) Doesn't change

Quick Quiz

Suppose you double the amplitude of the motion, what happens to the the total energy?

a) Doubles
b) Four times larger
c) Doesn't change

Energy

Since we know the total energy of a SHM, we can calculate the or displacement velocity at any point in time:

$$E_{Tot} = 1/2kA^2 = K+U = 1/2mv^2 + 1/2kx^2$$

So, if x=0, all E is in kinetic, and v is at max.

If x=A, all E is in potential, and v is zero.

Example

A 100g block is attached to a spring with spring constant 20 N/m. When the block is5cm from the equilibrium position, it moves with a speed of 1.5m/s.

a) What is the total energy of the system ?

b) What is the amplitude of the oscillations ?

Example

A 500g block on a spring is pulled 20cm and released. The motion has a period of 0.8s.

What is the velocity when the block is 15.4cm from the equilibrium position?