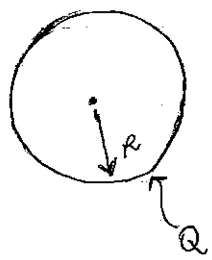
Problem 1



Electric field : the contribution to $\vec{E}$ by diametrically opposite charge elements cancel out.

$\therefore$ By symmetry , $\vec{E}(\vec{r}=0) = 0$

Potential :
$$V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi}\left(\frac{Q}{2\pi R}\right)\frac{R\,d\theta}{R}$$

$$\therefore \quad V = \frac{Q}{4\pi\epsilon_0 R}$$

Problem 2

Since the electric field inside the conducting shell is zero, the change in the energy of the system is due to the absence of the field within the shell, when the charge is at the centre of the shell :

$\therefore$ let inner and outer radii be $R-t$ and R.

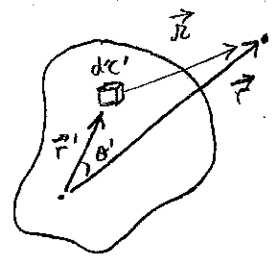
and
$$\Delta W = W_f - W_i = -\frac{\epsilon_0}{2}\int_{\substack{\text{inside}\\ \text{shell}}} E^2 d\tau$$

$$= -\frac{\epsilon_0}{2}\int_{\substack{\text{inside}\\ \text{shell}}} \frac{q^2}{(4\pi\epsilon_0 r^2)^2}\,(r^2\sin\theta\,dr\,d\theta\,d\phi)$$

$$= -\frac{q^2}{8\pi\epsilon_0}\int_{R-t}^{R}\frac{dr}{r^2} = -\frac{q^2}{8\pi\epsilon_0}\left(-\frac{1}{r}\right)_{R-t}^{R}$$

$$= \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{R-t} - \frac{1}{R}\right) = \frac{q^2}{8\pi\epsilon_0}\left[\frac{t}{R(R-t)}\right].$$

Problem 3



$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{r}\,')}{\mathscr{r}}\,d\tau'$$

Use $\displaystyle \frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$

$$\Rightarrow V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos\theta')\,\rho(\vec{r}\,')\,d\tau'$$

let $\rho\, d\tau \rightarrow \lambda\, dz = \frac{Q}{2a}\,dz$ and $r' \rightarrow z$

then $$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\underbrace{\int_{-a}^{a} z^n P_n(\cos\theta')\,\frac{Q}{2a}\,dz}_{I}$$ (drop prime on θ')

Integral $$I = \frac{Q}{2a}P_n(\cos\theta)\int_{-a}^{a} z^n\,dz = \frac{Q}{2a}P_n(\cos\theta)\,\frac{2a^{n+1}}{n+1}$$ for even n

(= 0 for odd n)

$$\therefore V = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}\sum_{n\text{ even}}\left[\frac{1}{n+1}\left(\frac{a}{r}\right)^n P_n(\cos\theta)\right]$$

$$\sim \underbrace{\frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r}\right)}_{\text{monopole}} + \underbrace{\frac{Q}{4\pi\epsilon_0}\left(\frac{a^2}{r^3}\right)\left[\frac{3\cos^2\theta - 1}{6}\right]}_{\text{quadrupole}} + \cdots$$

When $r \gg a$, quadrupole contribution is negligible

$\Rightarrow V \sim \dfrac{Q}{4\pi\epsilon_0 r}$ ∴ due to point charge Q, as expected.