Arts & Science 2D06

Mid-Year Exam **2019 December** **Name:** Solutions

Time allowed: 2.5 hours. No books or notes allowed. An electronic calculator may be used. *Complete solutions must be shown to obtain full marks for any of the problems.*

Formulae:

Solution for quadratic equation: $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Taylor series: $(1+x)^a \simeq 1 + ax$ for small x

Constant acceleration: $x = x_0 + v_0 t + \frac{1}{2}at^2$, $v = v_0 + at$, $v^2 = v_0^2 + 2ax$

$\sum F = ma$ $F_{AB} = -F_{BA}$ unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$

Kinetic friction: force $f = \mu_k N$ Static friction: force $f \leq \mu_s N$

Momentum: $\mathbf{p} = m\mathbf{v}$ Kinetic energy: $K = (1/2)mv^2$

Centripetal $a_c = v^2/r$ Rotational kinetic energy: $K = (1/2)I\omega^2$

Moment of inertia of disk/cylinder: $I = (1/2)MR^2$

Gravitational potential energy: $U = mgy$ Spring potential energy: $U = (1/2)kx^2$

Elastic collisions: $v_1 = \frac{(m_1 - m_2)}{(m_1 + m_2)}u_1$, $v_2 = \frac{2m_1}{(m_1 + m_2)}u_1$

Newton's universal law of gravity: $F_g = GMm/r^2$

Gamma factor: $\gamma = (1 - v^2/c^2)^{-1/2}$

Lorentz transformation: $x' = \gamma(x - vt)$, $t' = \gamma(t - \frac{v}{c^2}x)$.

Reverse Lorentz tranformation: $x = \gamma(x' + vt')$, $t = \gamma(t' + \frac{v}{c^2}x')$

Velocity addition: $u' = \frac{(u-v)}{(1 - uv/c^2)}$ Momentum: $p = \gamma mv$

Rest-mass energy: $E = mc^2$ Kinetic energy: $K = (\gamma - 1)mc^2$

Numerical Constants:

$c = 300,000$ km/sec $= 3.00 \times 10^8$ m/sec $= 1$ light-year/year (speed of light)

1 light-year (ly) $= 9.46 \times 10^{15}$ m

$G = 6.67 \times 10^{-11}$ N m^2/kg^2 (Newton's law of gravity constant)

$g = 9.8$ m/s^2 (acceleration of gravity near surface of Earth)

$M_E = 5.98 \times 10^{24}$ kg (mass of Earth)

$R_E = 6.38 \times 10^3$ km (radius of Earth)

PART A: Do all of the following short questions.

A1. [3] A bicycle's momentum is given by $\mathbf{p} = 8.5t\,\mathbf{i} - 4.6t^3\,\mathbf{j} - 1.9\,\mathbf{k}$. Find the force on the bicycle, as a function of time.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(8.5t\,\hat{i} - 4.6t^3\,\hat{j} - 1.9\,\hat{k})$$

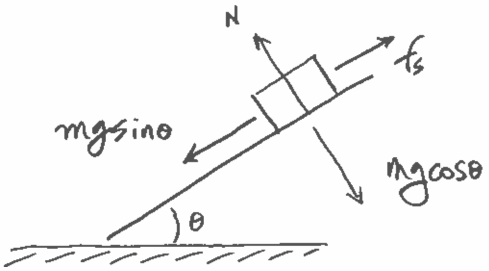
$$= 8.5\,\hat{i} - 13.8t^2\,\hat{j}$$

A2. [3] If Earth's mass and radius were both made two times bigger, by how much would the gravitational acceleration on its surface change?

$$g = \frac{GM_E}{R_E^2}$$

$$g_{new} = \frac{GM_{new}}{R_{new}^2} = \frac{G(2M_E)}{(2R_E)^2} = \frac{2}{4}\frac{GM_E}{R_E^2} = \frac{g}{2}$$

A3. [3] If the coefficient of static friction between tire rubber and a typical road pavement is 0.88, on how steep a road (*i.e.*, the largest angle) can you leave your car parked safely?



$$mg\sin\theta = f_s = \mu_s N = \mu_s (mg\cos\theta)$$

$$\Rightarrow \tan\theta = \mu_s$$

$$\theta = \tan^{-1}(0.88)$$

$$\therefore \quad \theta = 41^\circ$$

A4. [3] A particle of mass m travels at speed $v = 0.34$ c. If we want its momentum to be three times bigger, what should the particle's new speed be? Treat the question relativistically.

$$P_0 = \gamma_0 m V = \frac{mV}{\sqrt{1-V^2/c^2}}$$

$$P_f = \gamma_f m V_f = \frac{\cancel{m} V_f}{\sqrt{1-\frac{V_f^2}{c^2}}} = \frac{3\cancel{m}V}{\sqrt{1-V^2/c^2}}$$

$$\frac{V_f}{\sqrt{1-V_f^2/c^2}} = \frac{3(0.34c)}{\sqrt{1-\frac{0.34c^2}{c^2}}} = 1.08\,c$$

$$V_f^2 = 1.18c^2\left(1-\frac{V_f^2}{c^2}\right) = 1.18c^2 - 1.18\,V_f^2$$

$$V_f^2 = \frac{1.18c^2}{2.18} \Rightarrow V_f = 0.74c$$

A5. [3] A merry-go-round has a mass of 1500 kg and a radius of 8.0 m. Calculate how much work is needed to accelerate it from rest to a rotation rate of one revolution per 7.0 s. (For simplicity, you may model the merry-go-around as a solid cylinder. Also, ignore friction.)

$\omega_0 = 0$; $\omega_f = \dfrac{2\pi \text{ rad}}{7 \text{ sec}} = 0.9 \text{ s}^{-1}$

$\therefore \quad W = K_f - K_i = \frac{1}{2} I \omega_f^2$

$= \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega_f^2$

$= \frac{1}{2}\left(\frac{1}{2} \cdot 1500 \cdot 8^2\right)(0.9)^2$

$= 19\,440 \text{ J}$

A6. [3] Choose a topic or subtopic from Term 1 that you found especially interesting; it can be from the physics content or the historical/philosophical material. Using just the space below, explain your choice.

A7. [3] A soccer ball is kicked from a spot on a flat field with a speed of 20.0 m/s, and at an angle of 40° relative to the ground. For how long is the ball in flight before touching ground again? (Ignore air resistance.)

$$\Delta y = V_{0y} t - \frac{g t^2}{2} = 0$$

$$\Rightarrow \quad \frac{9.8 t}{2} = V_{0y} = V_0 \sin\theta$$

$$\therefore \quad t = \frac{2}{9.8} \cdot 20 \cdot \sin 40^\circ$$

$$\Rightarrow \quad t = 2.6 \text{ seconds.}$$

PART B: Do ANY 3 of the following 5 questions (your choice; 5 marks each; please indicate clearly your choices – otherwise, the first three attempted ones will be marked).

B1. [5] A pebble is thrown straight up with a speed of 10 m/s. After 1 second, another pebble is also thrown straight up, from the same height, with a speed of 16 m/s.

(a) Assuming they travel along the same path, find the time at which the two pebbles will collide.

$$\left.\begin{aligned} y_1 &= 10t - 4.9t^2 \\ y_2 &= 16(t-1) - 4.9(t-1)^2 \end{aligned}\right\} \quad \begin{aligned} y_1 &= y_2 \\ 10t - 4.9t^2 &= 16(t-1) - 4.9(t-1)^2 \end{aligned}$$

$$10t - \cancel{4.9t^2} = 16t - 16 - \cancel{4.9t^2} + 9.8t - 4.9$$

$$15.8t = 20.9$$

$$\underline{t = 1.32 \text{ sec}}$$

(b) At what height will the collision happen? (You may assume for simplicity that the pebbles are thrown from ground level.)

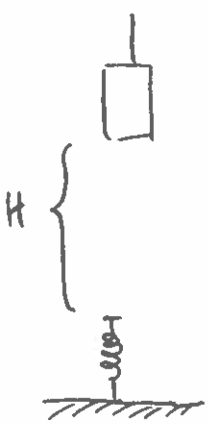
$$y_1 = 10(1.32) - 4.9(1.32)^2$$

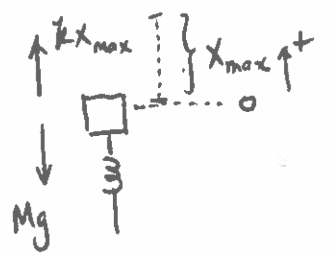
$$= \underline{4.66 \text{ m}}$$

B2. [5] Suppose you are designing a safety spring to be placed vertically at the bottom of an elevator shaft. If the elevator cable breaks when the elevator is at a height H above the top of the spring, what should the spring constant k be so that the passengers experience a deceleration no greater than four times g, as the elevator compresses the spring and slows down? Express your answer in terms of g, H, and M (the total mass of the elevator and passengers).

(*Hint*: the deceleration has its largest value of $4g$ when the spring force on the elevator also reaches its largest value during the spring's compression.)

largest spring force $\rightarrow$ x is max. $\rightarrow$ $V=0$





$$\therefore \quad M(4g) = kx_{max} - Mg$$

$$\Rightarrow x_{max} = \frac{5Mg}{k}$$

Energy Conservation :

$$E_0 = E_f$$

$$Mg(H + x_{max}) = \frac{1}{2} k x_{max}^2$$

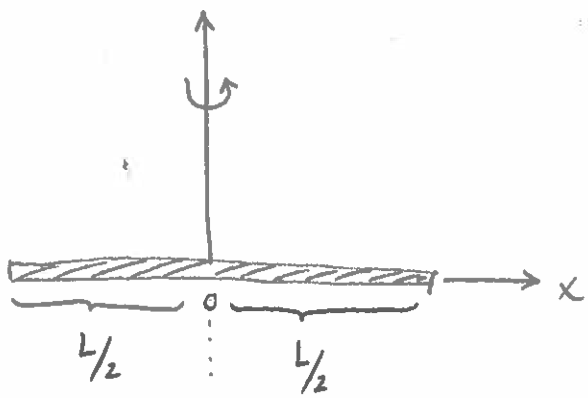
$$MgH + Mg\left(\frac{5Mg}{k}\right) = \frac{1}{2} k \left(\frac{5Mg}{k}\right)^2$$

$$\cancel{Mg}H + \frac{5(Mg)^{\cancel{2}}}{k} = \frac{25(Mg)^{\cancel{2}}}{2k}$$

$$Hk = \left(\frac{25}{2} - 5\right) Mg$$

$$\boxed{k = \frac{15Mg}{2H}}$$

B3. [5] A thin wire of length L lies horizontally along the x-axis, with its geometric centre at $x = 0$. Suppose that the wire's linear density λ (*i.e.*, mass per unit length) is given by $\lambda(x) = Ax^2 + B$, where A and B are constants. Find the wire's moment of inertia for rotation about an axis through $x = 0$ and perpendicular to the wire.



$$I = \int r^2 dm \quad , \quad dm = (Ax^2+B)\,dx$$

$$= \int_{-L/2}^{L/2} x^2 (Ax^2+B)\,dx$$

$$= \int_{-L/2}^{L/2} (Ax^4 + Bx^2)\,dx$$

$$= 2\int_{0}^{L/2} (Ax^4 + Bx^2)\,dx$$

$$\therefore\ I = 2\left[\frac{Ax^5}{5} + \frac{Bx^3}{3}\right]\Bigg|_0^{L/2}$$

$$= 2\left[\frac{AL^5}{160} + \frac{BL^3}{24}\right]$$

$$= \frac{AL^5}{80} + \frac{BL^3}{12}$$

B4. [5] (a) How much energy is contained in 0.5 gram of sand? If this energy is used to raise an object to a height of 1 km above Earth's surface, what would be this object's mass? (You may assume that g does not vary significantly over this distance.)

rest energy : $E = m_0 c^2 = (0.5\times10^{-3})(3\times10^{8})^2 = 4.5\times10^{13}\,\text{J}$

$$U = mgh \rightarrow m = \frac{U}{gh} = \frac{4.5\times10^{13}}{9.81\times1000}$$

$$\Rightarrow \quad m = 4.6\times10^{9}\ \text{kg}$$

(b) Suppose that a spaceship's engine generates energy for propulsion by conversion of mass. The passengers want to travel to a distant planet at a gentle speed of 0.8 c. If before take-off the spaceship has a mass of 200,000 kg, estimate how much of the spaceship's mass will be converted to energy in order to accelerate the spaceship from rest to this speed.

$$V = 0.8c \ : \quad \gamma = \frac{1}{\sqrt{1-\frac{0.64c^2}{c^2}}} = 1.7$$

$$\therefore \quad K = (\gamma-1)mc^2 = 0.7mc^2$$

where $m \cong 200000 - \Delta m$

So, $\quad \Delta m\cancel{c^2} \cong 0.7(200{,}000-\Delta m)\cancel{c^2}$

$$1.7\,\Delta m \cong 140{,}000$$

$$\underline{\Delta m \cong 82{,}000\ \text{kg}}$$

B5. [5] In developing a physical and foundational understanding of movement/motion this Term, we studied the ideas of Galileo, Newton and Einstein (among others). Choose two of these three scientists and discuss some key aspects of their contributions to our theory of motion, supplying supporting examples where appropriate.

Marks awarded will depend on content, organization, grammar, clarity, and the quality of your writing.

- Galileo: law of inertia; kinematics of free fall; role of friction ...

- Newton: laws of motion; dynamics; conservation laws ...

- Einstein: "absoluteness" of speed of light; relativization of space, time; new expressions for $\vec{p}$, E ...

36 marks total

36

(Happy Holidays!)

Extra Page for Scratch Work