Arts & Science 2D06

Mid-Year Exam **2014 December** **Name:** Solutions

Time allowed: 3 hours. No books or notes allowed. An electronic calculator may be used. *Complete solutions must be shown to obtain full marks for any of the problems.*

Formulae:

Solution for quadratic equation: $x = (-b \pm \sqrt{b^2 - 4ac})/2a$

Taylor series: $(1+x)^a \simeq 1 + ax$ for small x

Constant acceleration: $x = x_0 + v_0 t + \frac{1}{2}at^2$, $v = v_0 + at$, $v^2 = v_0^2 + 2ax$

$\sum F = ma$ $F_{AB} = -F_{BA}$ unit vectors $(\mathbf{i}, \mathbf{j}, \mathbf{k})$

Kinetic friction: force $f = \mu_k N$ Static friction: force $f \leq \mu_s N$

Momentum: $\mathbf{p} = m\mathbf{v}$ Kinetic Energy: $K = (1/2)mv^2$

Rotational Kinetic Energy: $K = (1/2)I\omega^2$

Moment of inertia of hoop: $I = MR^2$ Moment of inertia of sphere: $I = (2/5)MR^2$

Gravitational Potential: $U = mgy$ Spring potential: $U = (1/2)kx^2$

Elastic collisions: $v_1 = \frac{(m_1 - m_2)}{(m_1+m_2)}u_1$, $v_2 = \frac{2m_1}{(m_1+m_2)}u_1$

Centripetal acceleration: $a_c = v^2/r$

Newton's universal law of gravity: $F_g = GMm/r^2$

Gamma factor: $\gamma = (1 - v^2/c^2)^{-1/2}$

Lorentz transformation: $x' = \gamma(x - vt)$, $t' = \gamma(t - \frac{v}{c^2}x)$.

Reverse Lorentz tranformation: $x = \gamma(x' + vt')$, $t = \gamma(t' + \frac{v}{c^2}x')$

Velocity addition: $u' = \frac{(u-v)}{(1-uv/c^2)}$ Momentum: $p = \gamma mv$

Rest-mass Energy: $E = mc^2$ Kinetic energy: $K = (\gamma - 1)mc^2$

Total Energy: $\sqrt{p^2c^2 + m^2c^4}$

Numerical Constants:

$c = 300,000$ km/sec $= 3.00 \times 10^8$ m/sec (speed of light)

$G = 6.67 \times 10^{-11}$ N m^2/kg^2 (Newton's law of gravity constant)

$g = 9.8$ m/s^2 (acceleration of gravity near surface of Earth)

$M_E = 5.98 \times 10^{24}$ kg (mass of Earth)

PART A: Do all of the following short questions.

A1. [3] At what speed is a particle's relativistic momentum two times larger than its classical momentum? (Express your answer in terms of c.)

- rel. momentum : $p_{rel} = \gamma m v$, $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$
- classical momentum : $p_{cl} = mv$

$$\therefore \quad p_{rel} = 2 p_{cl}$$

$$\gamma \cancel{mv} = 2 \cancel{mv}$$

$$\frac{1}{\sqrt{1 - v^2/c^2}} = 2$$

$$1 - \frac{v^2}{c^2} = \frac{1}{4}$$

$$v^2 = \frac{3}{4} c^2$$

$$\rightarrow \underline{v = 0.87c}$$

A2. [3] An object's angular acceleration is given by $\alpha = 10t - 4t^3$ rad/s^2. Find the object's angular speed as a function of time, if at $t = 1$ second, its angular speed is measured to be 8 rad/s.

- angular speed : $\omega = \int \alpha \, dt'$

$$= \int (10t' - 4t'^3) \, dt'$$

$$= 5t^2 - t^4 + C$$

Now $\omega(t=1) = 8 = 5(1) - (1) + C$

$$\therefore C = 4$$

and $\underline{\omega(t) = 5t^2 - t^4 + 4}$

A3. [3] A rocket rises vertically from the ground (from rest) with an acceleration of 3 m/s^2. If it runs out of fuel after 6 seconds, what is the maximum height that it will reach?

• from start : $y = 1.5t^2$ and $V = 3t$

• after 6 seconds : $y = (1.5)(36) = 54\ m$; $V = 18\ m/s$

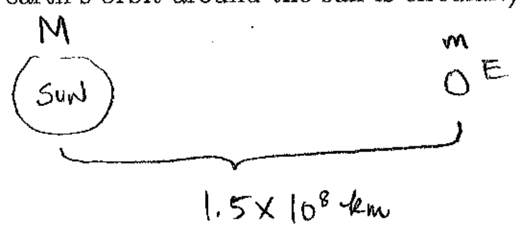
$\therefore$ after fuel runs out : $y = 54 + 18t - \frac{9.81}{2}t^2$

$V = 18 - 9.81\,t$

@ maximum height, $V = 0 \Rightarrow t = \frac{18}{9.81} = 1.8\ s$

$\therefore\ y_{max} = 54 + 18(1.8) - 4.9(1.8)^2 = \underline{70.5\,m}$

A4. [3] Find the mass of the sun, given that the distance between the sun and the earth is about 1.5 $\times$ 10^8 km. (*Hint*: the earth takes 1 year to go once around the sun; also, assume the earth's orbit around the sun is circular.)



$$F_g = \frac{mv^2}{r}$$

$$\frac{GM\cancel{m}}{r^2} = \frac{\cancel{m}V^2}{r}$$

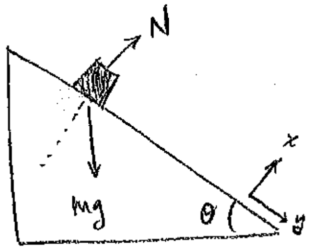
$$\therefore M = \frac{V^2 r}{G}$$

$$V = \frac{2\pi(1.5\times10^8)\,km\left(1000\ m/km\right)}{365\ days\left(\frac{24\ hr}{day}\right)\left(\frac{3600\ s}{hr}\right)} \cong 30\,000\ m/s$$

$$\therefore M = \frac{(30000)^2(1.5\times10^{11})}{6.67\times10^{-11}}$$

$$= \underline{2\times10^{30}\ kg}$$

A5. [3] A block is released at the top of a 25° incline. Using Newton's 2^{nd} law, calculate the coefficient of kinetic friction if it slides 2.4 meters down the incline in 4 seconds.



$$\sum_i^+ F_y = ma_y$$

$$mg\sin\theta - f = ma_y$$

$$mg\sin\theta - N\mu_k = ma_y$$

$$mg\sin\theta - mg\cos\theta\,\mu_k = ma_y$$

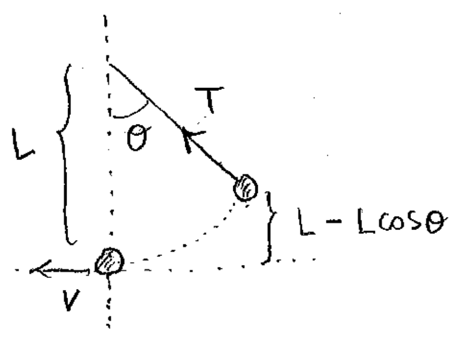
$$\therefore \mu_k = \frac{g\sin\theta - a_y}{g\cos\theta} \Rightarrow a_y \text{ is constant}$$

$$\therefore 2.4 = \frac{a_y(4)^2}{2}$$

$$a_y = 0.3\ m/s^2$$

$$\mu_k = \frac{(9.8)\sin 25° - 0.3}{9.8\cos 25°} = \underline{0.43}$$

A6. [3] A 1-m long pendulum is released from rest at an angle of 45°. The mass of the pendulum's bob is 100 grams. What is the pendulum's velocity at the lowest point? Also, find the tension in the string at this point. (Ignore the mass of the string.)



$$E_0 = E_f$$

$$mgh = \frac{1}{2}mv^2$$

$$\cancel{m}gL(1-\cos\theta) = \frac{1}{2}\cancel{m}v^2$$

$$v = \sqrt{2gL(1-\cos\theta)} = \sqrt{2(9.8)(1)(1-\cos 45°)}$$

$$= \underline{2.4\ m/s}$$

Tension :
(at bottom)

$$T - mg = \frac{mv^2}{L}$$

$$T = \frac{mv^2}{L} + mg = 0.1\left(\frac{(2.4)^2}{1} + 9.81\right) = \underline{1.56\ N}$$

A7. [3] A futuristic plane travels at $0.3c$ between two cities that are separated by 400 km. What are the duration of the trip and the distance traveled, as measured by the plane's pilot?

- duration of trip in lab frame : $\Delta t = \dfrac{400\,000\text{ m}}{(0.3)(3\times10^8\text{ m/s})} = \underline{4.4\times10^{-3}\text{ s}}$

- duration of trip in plane's frame :

$$\Delta t' = \frac{\Delta t}{\gamma} = (4.4\times10^{-3})\sqrt{1-\frac{(0.3c)^2}{c^2}} = \underline{4.2\times10^{-3}\text{ s}}$$

- distance travelled according to pilot :

$$L = (0.3c)(4.2\times10^{-3}\text{ s}) = \underline{378\text{ km}}$$

A8. [3] Suppose that an external force acting on a spring is given by $F(x) = 8x + 0.4x^2$ N, where x is the distance that the spring is stretched from its equilibrium configuration. With this force, what is the external work needed to stretch the spring by 2 meters?

$$W = \int_0^2 F(x)\,dx = \int_0^2 (8x + 0.4x^2)\,dx = \left.\frac{8x^2}{2} + \frac{0.4x^3}{3}\right|_0^2$$

$$= 16 + 1.1 = \underline{17.1\text{ J}}$$

PART B: Do ANY 4 of the following 7 questions (your choice; 5 marks each).

B1. [5] A 25-g block on a frictionless table is firmly attached to one end of a spring with $k = 18$ N/m, with the other end anchored to the wall. A 25-g ball is thrown horizontally toward the block with a speed of 7 m/s.

(a) If the collision is perfectly elastic, calculate the ball's speed immediately after the collision.

(b) Find the maximum compression of the spring.

(c) Repeat parts (a) and (b) for a perfectly inelastic collision.

(a) perfectly elastic, same mass $\Rightarrow$ $\underline{V_{ball} = 0}$

(b) $\cancel{\frac{1}{2}} m_{block} V^2 = \cancel{\frac{1}{2}} k x_{max}^2 \quad \therefore \quad X_{max} = \sqrt{\frac{m_{block} V^2}{k}}$

$$= \sqrt{\frac{(0.025)(7)^2}{18}}$$

$$= \underline{0.26\,m}$$

(c) perfectly inelastic collision :

$$\vec{P}_{before} = \vec{P}_{after}$$

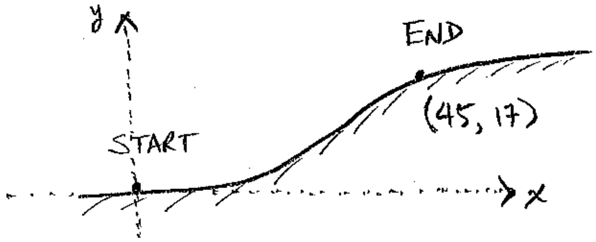
$$(0.025)(7) = (0.025 + 0.025)V$$

$$\underline{V = 3.5\,m/s}$$

$$\therefore \quad \frac{1}{2} m V^2 = \frac{1}{2} k x_{max}^2$$

$$x_{max} = \sqrt{\frac{(0.050)(3.5)^2}{18}} = \underline{0.18\,m.}$$

B2. [5] A golf ball, starting from level ground, is hit at an initial angle of 30° and lands on a green as shown, at $(x, y) = (45, 17)$ meters relative to its starting location. What was its initial speed?



$$\vec{V_0} = V_{0x}\,\hat{i} + V_{0y}\,\hat{j} \longrightarrow V_{0x} = V_0\cos\theta \;;\; V_{0y} = V_0\sin\theta$$

- X-direction : $X = X_0 + V_{0x} t$

$$\therefore\; X = (V_0\cos\theta)t \qquad \text{------} \quad (1)$$

- y-direction : $y = y_0 + V_{0y}t - \frac{g}{2}t^2$

$$\therefore\; y = (V_0\sin\theta)t - \frac{g}{2}t^2 \qquad \text{------} \quad (2)$$

Sub (1) in (2) to eliminate t :

$$y = (\cancel{V_0}\sin\theta)\left[\frac{x}{\cancel{V_0}\cos\theta}\right] - \frac{g}{2}\left[\frac{x}{V_0\cos\theta}\right]^2$$

$$= x\tan\theta - \frac{g}{2}\,\frac{x^2}{V_0^2\cos^2\theta}$$

$$17 = 45\tan 30° - \frac{9.81}{2}\,\frac{(45)^2}{V_0^2\cos^2 30°}$$

$$17 = 26 - \frac{13244}{V_0^2}$$

$$9V_0^2 = 13244 \Longrightarrow V_0^2 = 1472 \qquad \therefore\; \underline{V_0 = 38.4\ m/s}$$

B3. [5] A late riser runs at a constant 4.5 m/s to catch a bus. He reaches the bus stop 2 seconds after the bus starts from rest with an acceleration of 1 m/s^2.

(a) Where and when does the person catch the bus?

(b) How much longer could he have slept in and still made it to work on time?

(a) Bus: $x_B = x_{0B} + V_{0B}t_B + \frac{1}{2}a_B t_B^2$

$= \frac{1}{2}t_B^2$

Person: $x_p = x_{0p} + V_p t_p = 4.5 t_p$

Also, $t_B = t_p + 2$

$\therefore$ bus is caught when: $x_B = x_p$

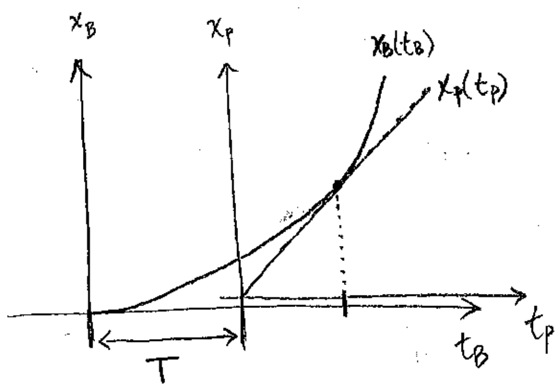
$$\frac{(t_p+2)^2}{2} = 4.5 t_p$$

$$t_p^2 - 5t_p + 4 = 0$$

$\underline{t_p = 1 \text{ sec}}$ (2^{nd} solution is larger → ignore)

and $\underline{x_p = 4.5 \text{ m}}$ (from stop)

(b)



find this; compare to $\Delta t = 2$s from (a).

$$\frac{(t_p+T)^2}{2} = 4.5 t_p$$

$$t_p^2 + 2T t_p + T^2 = 9 t_p$$

$$t_p^2 + (2T-9)t_p + T^2 = 0$$

$$\therefore \quad t_p = \frac{(9-2T) \pm \sqrt{(2T-9)^2 - 4T^2}}{2}$$

require one solution:

$\Rightarrow (2T-9)^2 - 4T^2 = 0$

$-36T + 81 = 0$

$T = 2.25$ seconds

$\therefore$ he could have slept in by an extra $\underline{0.25 \text{ sec.}}$

B4. [5] A solid sphere of radius R is placed at a height of 30 cm on a 15° slope. It is then released, and rolls (without slipping) to the bottom of the slope.

(a) From what height should a circular hoop of radius R be released on the same slope in order to end up the sphere's speed at the bottom?

(b) Can a circular hoop of a different diameter be released from a height of 30 cm and end up with the sphere's speed at the bottom? If so, what is the diameter? If not, why not?

(a) Sphere:

$$E_0 = E_f$$

$$Mgy_0 + \cancel{\tfrac{1}{2}MV_0^2}^{\,0} + \cancel{\tfrac{1}{2}I\omega_0^2}^{\,0} = \cancel{Mgy_f}^{\,0} + \tfrac{1}{2}MV_f^2 + \tfrac{1}{2}I\omega_f^2$$

$$\therefore\ \tfrac{1}{2}\cancel{M}V_f^2 + \tfrac{1}{\cancel{2}}\left(\tfrac{\cancel{2}}{5}\cancel{M}\cancel{R^2}\right)\left(\tfrac{V_f}{\cancel{R}}\right)^2 = \cancel{M}gy_0$$

$$\frac{V_f^2}{2} + \frac{V_f^2}{5} = gy_0$$

$$\frac{7V_f^2}{10} = gy_0 \Rightarrow V_f = \sqrt{\frac{10gy_0}{7}} = \sqrt{\frac{10(9.81)(0.3)}{7}}$$

$$= 2.1\ m/s$$

hoop:

$$\cancel{M}gy_0 = \tfrac{1}{2}\cancel{M}V_f^2 + \tfrac{1}{2}\cancel{(MR^2)}\left(\frac{V_f^2}{\cancel{R^2}}\right) = \cancel{M}V_f^2$$

$$y_0 = \frac{V_f^2}{g} = \frac{(2.1)^2}{9.81} = \underline{0.45\ m}$$

(b) for a hoop from 30 cm: $V_f = \sqrt{gy_0} = \sqrt{(9.81)(0.3)} = 1.7\ m/s$

which is independent of $R \Rightarrow$ <u>NO</u>.

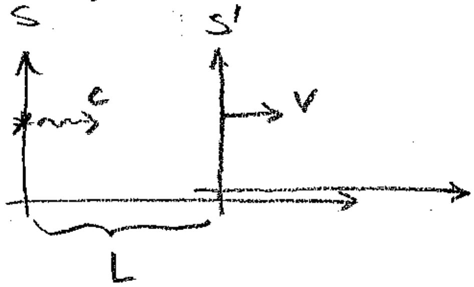
B5. [5] A light detector aboard a rocket-ship is moving relative to Earth at speed v along the positive x-axis. When the ship is at a distance $x = L$ from Earth, a light flash is emitted from a station on Earth.

(a) How long does it take for this flash to reach the detector, according to an observer on the rocket-ship?

(b) Repeat part (a) for an observer back on Earth.

(a) Let the origins of the frames (S, S') coincide at $t_1 = t_1' = 0$

when light flash is emitted:



$S: \quad x_2 = 0, \quad t_2 = \frac{L}{V}$

$S': \quad x_2' = \gamma(x_2 - Vt_2) = -\gamma L$

$\left(\text{and } t_2' = \gamma\left(t_2 - \frac{Vx_2}{c^2}\right) = \frac{\gamma L}{V}\right)$

$\therefore$ In S', light is emitted from a distance γL from ship.

- Note that detector is stationary wrt S'.

$\rightarrow$ Einstein's 2nd postulate: in S', the relative velocity between light and ship is $\underline{c}$.

$$\therefore \boxed{\Delta t' = \frac{\gamma L}{c}}$$

(b) in Earth's frame: $L + (\Delta t)V = c\Delta t$

$$(c - V)\Delta t = L$$

$$\boxed{\Delta t = \frac{L}{c - V}}$$

B6. [5] Albert Einstein once drew an analogy between the study of nature through physics, and the process of solving a grand detective story. Richard Feynman – another distinguished 20^{th} century physicist – in turn compared "doing physics" to discerning the rules of a gigantic board game by watching the pieces move. Choose one of these perspectives, and discuss why the analogy is appropriate and/or inadequate.

Marks awarded will depend on content, organization, grammar, clarity, and the quality of your writing.

Some thoughts (not exhaustive); some from your answers:

Einstein's view: Similarities w/ science/physics:

- In both cases, a puzzle needs to be solved.
- collection of data / evidence
- use of logic / reasoning

Differences:

- In science, one can repeat experiments
- Science is impersonal (for the most part).

Feynman's view:

Similarities: rules of game, rules of nature

observation; collection of information

"gigantic" → true of the universe

Differences: at any given time, some phenomena in science are unobservable.

passive observing of board game vs. active in experimentation.

B7. [5] Through his work, Isaac Newton was largely responsible for ushering in the scientific age, one characterized by a new worldview whose salient features include *universality*, *determinism*, and *reductionism*. Explain two of these features, using examples from this term's course material.

Marks awarded will depend on content, organization, grammar, clarity, and the quality of your writing.

- Universality : fundamental laws of nature are applicable at all places and times.

  e.g., Newton's law of (universal) gravitation, and many others.

- determinism : through mathematics, the laws make sharp predictions for observable/measurable quantities ; or, laws reveal tight links between cause and effect.

  e.g., $\vec{F} = m\vec{a}$, from which we can get $x(t)$, etc.

- reductionism : nature is such that many phenomena can be explained by a few underlying laws; or, the behaviour of the whole should be explained in terms of that of its constituent parts.

  e.g. , the wide applicability of Newton's laws to motions of all kinds.

44 marks total

(Happy Holidays!)