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UNCERTAINTIES

Since the quantity of interest in an experiment is rarely obtained by measuring that quantity directly, we must understand how error propagates when mathematical operations are performed on measured quantities. Suppose we have a simple experiment where we want to measure velocity, V, by measuring distance, d, and time, t. We take measurements and come up with measured quantities $d \pm \delta d$ and $t \pm \delta t$. We can easily estimate V by dividing d by t, but we also need to know how to find δV. Below we investigate how error propagates when mathematical operations are performed on quantities $x, \ldots, z$ and $u, \ldots, w$ that comprise the desired quantity q.

Uncertainty in Sums and Differences

Quantities $x, \ldots, w$ are measured with small uncertainties $\delta x, \ldots, \delta w$ and

$$q = x + \ldots + z - (u + \ldots + w)$$

Then the absolute uncertainty in the computed value of q is:

$$\delta q = \sqrt{(\delta x)^2 + \ldots + (\delta z)^2 + (\delta u)^2 + \ldots (\delta w)^2}$$

For simplicity one may use:

$$\delta q \approx \delta x + \ldots + \delta z + (\delta u + \ldots + \delta w)$$

In summery, when adding or subtracting quantities, their uncertainties add.

Example:

If $$q = c_1 x + \ldots + c_n z$$

Where $c_1, \ldots, c_n$ are constants and $x, \ldots, z$ are measured quantities. Then the uncertainty in the computed value of q is given by:

$$\delta q = \sqrt{c_1^2 (\delta x)^2 + \ldots + c_n^2 (\delta z)^2}$$

Uncertainty in Products and Quotients

Quantities $x, \ldots, w$ are measured with small uncertainties $\delta x, \ldots, \delta w$ and where $c_1, \ldots, c_n$ and $d_1, \ldots, d_m$ are constants, and

$$q = \frac{c_1 x \times \ldots \times c_n z}{d_1 u \times \ldots \times d_m w}$$

Then the fractional (relative) uncertainty in the computed value of q is:

$$\frac{\delta q}{|q|} = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \ldots + \left(\frac{\delta z}{z}\right)^2 + \left(\frac{\delta u}{u}\right)^2 + \ldots + \left(\frac{\delta w}{w}\right)^2}$$

For simplicity one may use:

$$\frac{\delta q}{|q|} \approx \frac{\delta x}{|x|} + \ldots + \frac{\delta z}{|z|} + \frac{\delta u}{|u|} + \ldots + \frac{\delta w}{|w|}$$

In summary, when any number of quantities are multiplied or divided, their *relative (fractional)* uncertainties add.

Example:

If

$$q = \frac{cx^n}{z^m} = cx^n z^{-m}$$

Where c is a constant, n and m are exponents that can be fractional if representing a root, and may be positive or negative, and x and z are measured quantities, then the relative (fractional) uncertainty is given by:

$$\frac{\delta q}{|q|} = |n|\frac{\delta x}{|x|} + |m|\frac{\delta z}{|z|}$$

NOTE: The absolute value of the negative exponent is used in the fractional uncertainty equation.

Uncertainty in a Function of Several Variables

If $x, \ldots, z$ are measured with independent and random uncertainties $\delta x, \ldots, \delta z$ and are used to compute $q(x, \ldots, z)$ then the uncertainty in q is:

$$\delta q = \sqrt{\left(\frac{\partial q}{\partial x}\delta x\right)^2 + \ldots + \left(\frac{\partial q}{\partial z}\delta z\right)^2}$$

More generally, we can write a formula for computing the propagated error in a function of several variables. We must compute partial derivatives (i.e. $\partial q/\partial x$) due to the fact that q can be a function of multiple variables.

Example:

If

$$q = \ln cx$$

Where c is a constant, and x is a measured quantity, then the uncertainty of q is given by:

$$\delta q = \frac{\delta x}{|x|}$$

For logarithms to base 10 it is necessary to convert to base e:

$$q = c\log_{10} x = (c \times 0.43)\ln x$$

Example:

If

$$q = e^x$$

Where x is a measured quantity, then the uncertainty of q is given by:

$$\delta q = e^x \times \delta x$$

Example:

If

$$q = \sin\theta$$

Where θ is a measured quantity and $\delta\theta$ must be in RADIANS. Then the uncertainty of q is given by:

$$\delta q = \cos\theta \times \delta\theta$$