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PHYS 4E03

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Assignment #8: Solutions

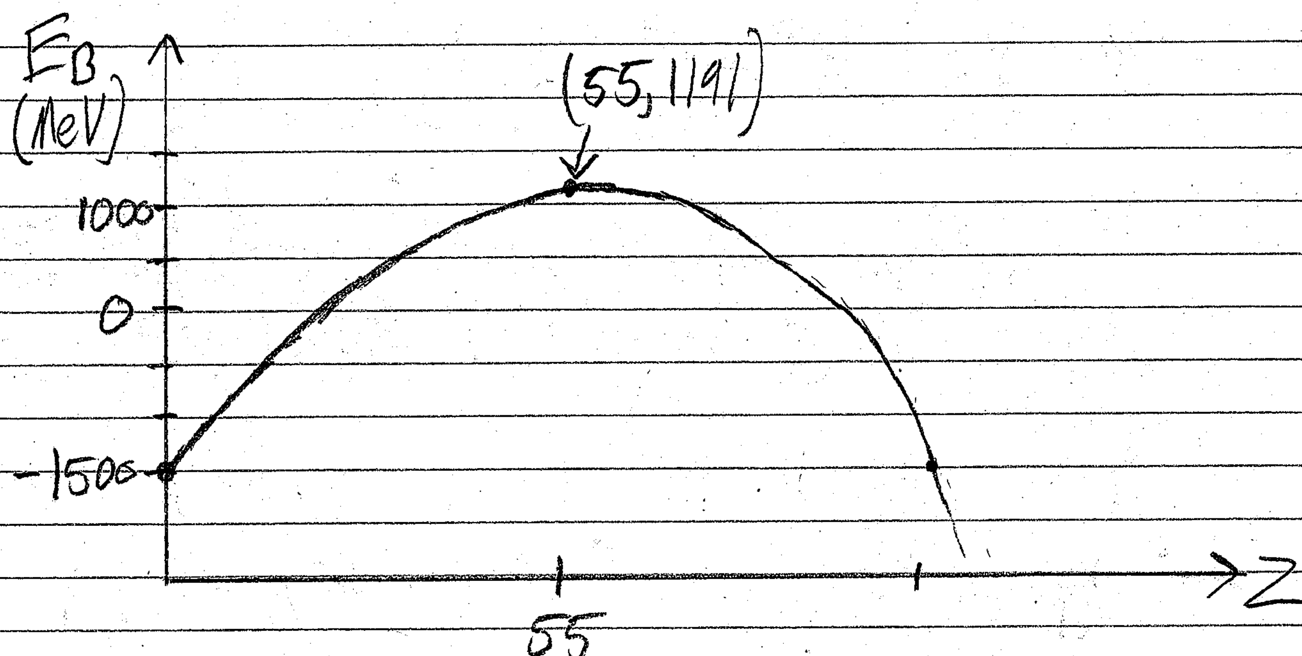
$\boxed{1}$
(5.1)

We want to find the integer Z for which $A = 127$ is stable. This occurs at the max of

$$E_B(Z,N) = C_v A - C_s A^{2/3} - C_c\left(\frac{Z(Z-1)}{A^{1/3}}\right) - C_{sym}\left[\frac{(N-Z)^2}{A}\right] + \frac{\Delta}{A^{1/2}}$$

Weizäcker mass formula

where the constant $C_{(\dots)}$ are given by (5.19)



$\Rightarrow$ Max(E_B) occurs at $\boxed{Z = 55}$

	E_B (MeV)	
For $Z=48$:	1144	$\lvert\Delta E\rvert = 12$ MeV
$Z=49$:	1156.2	
and $Z=57$:	1188.2	$\lvert\Delta E\rvert = 4$ MeV
$Z=58$:	1184	

The possible decays are:

① β^+: ${}^{A}_{Z}X \rightarrow {}^{A}_{Z-1}X + e^+ + \nu_e$

② β^-: ${}^{A}_{Z}X \rightarrow {}^{A}_{Z+1}X + e^- + \bar{\nu}_e$

③ α: ${}^{A}_{Z}X \rightarrow {}^{A-4}_{Z-2}X + {}^{4}_{2}He$

④ γ: ${}^{A}_{Z}X^* \rightarrow {}^{A}_{Z}X + \gamma$

$m_e \approx 0.5$ MeV

For ① or ②, we need $E_B(Z,A) < E_B(Z\mp 1,A) - m_e$

and for ③, we need $E_B(Z,A) < E_B(Z-2,A-4) - E_B(2,4)$

$\boxed{1}$ For $A=127$ and $Z=48$, $E_B(48,127) = \boxed{1144 \text{ MeV}}$

$\Rightarrow E_B(47,127) - m_e = 1129.5$ MeV

$\Rightarrow E_B(49,127) - m_e = \boxed{1155.7 \text{ MeV}}$

$\Rightarrow E_B(46,123) - E_B(2,4) = 1082.1$ MeV

$\Rightarrow$ Z = 48 will β^- decay to Z = 49

2) For $A = 127$ and $Z = 49$, $E_B(49,127) = \boxed{1156.2\ MeV}$

$\Rightarrow E_B(48,127) - m_e = 1143.5\ MeV$

$\Rightarrow E_B(50,127) - m_e = 1166\ MeV$

$\Rightarrow E_B(47,123) - E_B(2,4) = 1095.4\ MeV$

$\Rightarrow$ $Z = 49$ will β^- decay to $Z = 50$

3) For $A = 127$ and $Z = 57$, $E_B(57,127) = \boxed{1188.2\ MeV}$

$\Rightarrow E_B(56,127) - m_e = 1190\ MeV$

$\Rightarrow E_B(58,127) - m_e = 1183.5\ MeV$

$\Rightarrow E_B(55,123) - E_B(2,4) = 1134.5\ MeV$

$\Rightarrow$ $Z = 57$ will β^+ decay to $Z = 56$

4) For $A = 127$ and $Z = 58$, $E_B(58,127) = \boxed{1184\ MeV}$

$\Rightarrow E_B(57,127) - m_e = 1187.7\ MeV$

$\Rightarrow E_B(59,127) - m_e = 1177.6\ MeV$

$\Rightarrow E_B(56,123) - E_B(2,4) = 1131\ MeV$

$\Rightarrow$ $Z = 58$ will β^+ decay to $Z = 57$

The decay energies are:

$\boxed{1}$ $Q = 11.7$ MeV $\boxed{2}$ $Q = 9.8$ MeV

$\boxed{3}$ $Q = 1.8$ MeV $\boxed{4}$ $Q = 3.7$ MeV

We assume $\Gamma \simeq \frac{G_F^2 Q^5}{(4\pi)^3}$, $G_F = 1.1664 \times 10^{-5}$ GeV^{-2}

so $\tau = \frac{1}{\Gamma}$ and $\hbar = 1 = 6.582 \times 10^{-22}$ MeV·s

$\boxed{1}$ $\tau = 0.04$ s $\boxed{2}$ $\tau = 0.1$ s

$\boxed{3}$ $\tau = 508$ s $\boxed{4}$ $\tau = 14$ s

$\boxed{2}$ (5.3) We have $H_{so} = -\frac{\vec{S}\cdot\vec{L}}{mr^2}$ and want to find the A dependence of C_{so} in the expression

$E_{so} = -\frac{C_{so}}{2}\left[j(j+1) - \ell(\ell+1) - s(s+1)\right]$

We also want C_{so} such that for ^{5}He we have $\left|E_{so}(j=\frac{3}{2}) - E_{so}(j=\frac{1}{2})\right| = 4$ MeV

$\Rightarrow$ ^{5}He has $\ell = 1$ and we know that $s = \frac{1}{2}$

$\Rightarrow E_{so}(j=\frac{3}{2}) = -\frac{C_{so}}{2}$, $E_{so}(j=\frac{1}{2}) = C_{so}$

$$\Rightarrow \Delta E = \frac{3}{2} C_{so} \equiv 4\,\text{MeV} \Rightarrow \boxed{C_{so} = \frac{8}{3}\,\text{MeV}}$$

To get the A-dep., we have

$$H_{so}\psi = E_{so}\psi = -\frac{1}{mr^2}(\vec{S}\cdot\vec{L})\psi$$

$\Rightarrow$ Use $\vec{J} = \vec{S} + \vec{L}$, so $\vec{J}^2 = \vec{S}^2 + \vec{L}^2 + 2\vec{S}\cdot\vec{L}$

and thus $\vec{S}\cdot\vec{L} = \frac{1}{2}\left(\vec{J}^2 - \vec{L}^2 - \vec{S}^2\right)$

Use $\vec{J}^2\psi = j(j+1)\psi$, $\vec{L}^2 = l(l+1)\psi$, $\vec{S}^2\psi = s(s+1)\psi$

Thus, we get $\boxed{C_{so} = \frac{1}{mr^2}}$

Since m is the particle's mass, $m \sim A$

Also, the radial distance r can be associated to the nuclear radius which scales as $\sim A^{1/3}$

$$\Rightarrow \boxed{C_{so}} \sim \frac{1}{A}\cdot\frac{1}{A^{2/3}} \sim \boxed{A^{-5/3}}$$

3 (5.4) Isomers have a long half-life due to the excited state being stable. According to the shell model certain values of Z and N, so for $A = Z + N$ as well, lead to particularly stable nuclei because they are well bound. Thus, many nuclei with Z and A close to these "magic" values will all be isomers, hence the concept of "islands of isomerism".

For ${}^{85}_{38}Sr^*$ to decay to ${}^{85}_{38}Sr$ it must undergo gamma decay and emit a photon. This however means that $|\Delta j| = 1$, but ${}^{85}Sr^*$ has spin $\frac{1}{2}$ while ${}^{85}Sr$ has spin $\frac{9}{2}$ so $|\Delta j| = 4$. Thus, the gamma decay is "forbidden" by the high spin change involved, which lead to a longer half-life.

[4] (5.5) We want to use the expressions for matrix elements:

① $\langle f | \vec{S}^{(1)} | i \rangle = \frac{1}{2} \sum_{abcd} \langle ab | \vec{\sigma}_{ac} \delta_{bd} | cd \rangle$

② $\langle f | \vec{S}^{(2)} | i \rangle = \frac{1}{2} \sum_{abcd} \langle ab | \vec{\sigma}_{bd} \delta_{ac} | cd \rangle$

Also, $\sigma^1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\boxed{1}$ We want $S_x^{(1)} = \begin{array}{cc} \begin{pmatrix} \boxed{\ } & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} & \begin{matrix} \to \langle \uparrow\uparrow | \\ \to \langle \downarrow\uparrow | \\ \to \langle \uparrow\downarrow | \\ \to \langle \downarrow\downarrow | \end{matrix} \\ \begin{matrix} |\uparrow\uparrow\rangle & |\downarrow\uparrow\rangle & |\uparrow\downarrow\rangle & |\downarrow\downarrow\rangle \end{matrix} & \end{array}$

For example, the top left entry $\boxed{\ }$ is given by $\frac{\sigma^1_{ac} \delta_{bd}}{2}$ for $a=b=c=d=\uparrow$, so $\frac{\sigma^1_{\uparrow\uparrow} \cdot \delta_{\uparrow\uparrow}}{2} = 0$

Then 0 is $a=b=\uparrow$, $c=\downarrow$ and $d=\uparrow$, so this gives $\frac{1}{2} \cdot \sigma^1_{\uparrow\downarrow} \delta_{\uparrow\uparrow} = \frac{1}{2} \cdot 1 \cdot 1 = \frac{1}{2}$

Doing so for all $a,b,c,d = \{\uparrow, \downarrow\}$ gives us the required results for ① and ②.

We can calculate $\vec{S}^{(1)}\cdot\vec{S}^{(1)} = \left(S_x^{(1)}\right)^2 + \left(S_y^{(1)}\right)^2 + \left(S_z^{(1)}\right)^2$

$$\Rightarrow \left(\vec{S}^{(1)}\right)^2 = \frac{1}{4}\left[\begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix}\begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix} + \begin{pmatrix}\sigma_y & 0\\ 0 & \sigma_y\end{pmatrix}\begin{pmatrix}\sigma_y & 0\\ 0 & \sigma_y\end{pmatrix} + \begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix}\right]$$

$$= \frac{1}{4}\left[\begin{pmatrix}\sigma_x^2 & 0\\ 0 & \sigma_x^2\end{pmatrix} + \begin{pmatrix}\sigma_y^2 & 0\\ 0 & \sigma_y^2\end{pmatrix} + \begin{pmatrix}\sigma_z^2 & 0\\ 0 & \sigma_z^2\end{pmatrix}\right]$$

(But $\sigma_i^2 = I$)

$$= \frac{1}{4}\left(\begin{pmatrix}3I & 0\\ 0 & 3I\end{pmatrix}\right) = \boxed{\frac{3}{4} I}$$

Similarly for $\vec{S}^{(2)}\cdot\vec{S}^{(2)} = \boxed{\frac{3}{4} I}$

We get the eigenvalues of $(\vec{S})^2 = \left(\vec{S}^{(1)} + \vec{S}^{(2)}\right)^2$

using $\underbrace{\left((\vec{S}\cdot\vec{S}) - \lambda I\right)}_{(*)} \equiv 0$ and taking $\det(*) = 0$

$$\Rightarrow \vec{S}\cdot\vec{S} = \left(\vec{S}^{(1)}\right)^2 + \left(\vec{S}^{(2)}\right)^2 + \vec{S}^{(1)}\cdot\vec{S}^{(2)} + \vec{S}^{(2)}\cdot\vec{S}^{(1)}$$

$$= \begin{pmatrix}2 & 0 & 0 & 0\\ 0 & 1 & 1 & 0\\ 0 & 1 & 1 & 0\\ 0 & 0 & 0 & 2\end{pmatrix}$$

$$\Rightarrow \text{Det}\begin{pmatrix} 2-\lambda & 0 & 0 & 0 \\ 0 & 1-\lambda & 1 & 0 \\ 0 & 1 & 1-\lambda & 0 \\ 0 & 0 & 0 & 2-\lambda \end{pmatrix} = (2-\lambda)(2-\lambda)\left[(1-\lambda)^2 - 1\right]$$

$$= (2-\lambda)^2(1-2\lambda+\lambda^2-1) = (2-\lambda)^2(\lambda^2-2\lambda)$$

$$= (\lambda-2)^3\lambda \equiv 0$$

$\Rightarrow \boxed{\lambda = 0}$ or $\boxed{\underbrace{\lambda = 2}_{\text{threefold degenerate}}}$

$\boxed{5}$ 15.6) We want the explicit form of

$$\vec{T}^{(1)}\cdot\vec{T}^{(2)} = \frac{1}{4}\left(\tau_1^{(1)}\cdot\tau_1^{(2)} + \tau_2^{(1)}\cdot\tau_2^{(2)} + \tau_3^{(1)}\cdot\tau_3^{(2)}\right)$$

where $\tau_i^{(1)}, \tau_i^{(2)}$ are exactly the same as the $S^1_{(i)}, S^2_{(i)}$ from the previous question by comparing (5.30), (5.31) with (5.40), (5.41)

With basis like in $\boxed{4}$ but with $\uparrow \equiv p$, $\downarrow \equiv n$

$$\Rightarrow \vec{T}^{(1)}\cdot\vec{T}^{(2)} = \frac{1}{4}\left[\begin{pmatrix}\sigma_x & 0\\ 0 & \sigma_x\end{pmatrix}\begin{pmatrix}0 & I\\ I & 0\end{pmatrix} + \begin{pmatrix}\sigma_y & 0\\ 0 & \sigma_y\end{pmatrix}\begin{pmatrix}0 & -iI\\ iI & 0\end{pmatrix} + \begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix}\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}\right.$$

$$= \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We have $V = V_0 I \; V_1 \vec{T}^{(1)} \cdot \vec{T}^{(2)}$

$$= \begin{pmatrix} V_0 + \frac{V_1}{4} & 0 & 0 & 0 \\ 0 & V_0 - \frac{V_1}{4} & \frac{V_1}{2} & 0 \\ 0 & \frac{V_1}{2} & V_0 - \frac{V_1}{4} & 0 \\ 0 & 0 & 0 & V_0 + \frac{V_1}{4} \end{pmatrix}$$

Thus, $\langle nn | V | nn \rangle = V_0 + \frac{V_1}{4}$

$\langle np | V | pn \rangle = \frac{V_1}{2}$

$\langle nn | V | np \rangle = 0$

$\langle pp | V | pp \rangle = V_0 + \frac{V_1}{4}$

Exercise 5.7

$$\mathcal{M} = 16G_F^2C^2|V_{ud}|^2\left[2(k\cdot q)(r\cdot q) - q^2(k\cdot r)\right]$$

Three-body decay in rest-frame:

$$\begin{aligned}
\frac{d\Gamma}{d^3\boldsymbol{q}\cdot d^3\boldsymbol{k}\cdot d^3\boldsymbol{r}} &= \frac{1}{2E_A}\left[\frac{\mathcal{M}}{[(2\pi)^3 2E_B][(2\pi)^3 2E_e][(2\pi)^3 2E_\nu]}\right](2\pi)^4\delta^4(p-q-k-r) \\
&= \frac{16G_F^2C^2|V_{ud}|^2\left[2(k\cdot q)(r\cdot q) - q^2(k\cdot r)\right]}{16(2\pi)^5 E_A E_B E_e E_\nu}\delta^4(p-q-k-r) \\
\frac{d\Gamma}{d^3\boldsymbol{q}\cdot d^3\boldsymbol{k}} &= \int d^3\boldsymbol{r}\,\frac{G_F^2C^2|V_{ud}|^2\left[2(k\cdot q)(r\cdot q) - q^2(k\cdot r)\right]}{(2\pi)^5 E_A E_B E_e E_\nu}\delta^4(p-q-k-r) \\
&= \frac{G_F^2C^2|V_{ud}|^2\left[2(k\cdot q)(r\cdot q) - q^2(k\cdot r)\right]}{(2\pi)^5 E_A E_B E_e E_\nu}\delta(m_A - E_B - E_e - E_\nu)|_{0=\boldsymbol{q}+\boldsymbol{k}+\boldsymbol{r}} \\
\frac{d\Gamma}{d^3\boldsymbol{k}} &= \frac{G_F^2C^2|V_{ud}|^2}{(2\pi)^5}\int|\boldsymbol{q}|^2 d|\boldsymbol{q}|\int_0^{2\pi}d\phi\int_0^\pi \sin\theta d\theta\frac{\left[2(k\cdot q)(r\cdot q) - q^2(k\cdot r)\right]}{E_A E_B E_e E_\nu}\delta(m_A - E_B - E_e - E_\nu) \\
&= \frac{G_F^2C^2|V_{ud}|^2}{(2\pi)^4}\int|\boldsymbol{q}|^2 d|\boldsymbol{q}|\int_0^\pi \sin\theta d\theta\frac{\left[2(k\cdot q)(r\cdot q) - q^2(k\cdot r)\right]}{E_A E_B E_e E_\nu}\delta(m_A - E_B - E_e - E_\nu)
\end{aligned}$$

Where

$$\int dx\, f(x)\delta(g(x)) = \frac{f(x_0)}{|g'(x_0)|}\quad \text{such that } g(x_0) = 0$$

$$\begin{aligned}
E_e + E_\nu &= \sqrt{|\boldsymbol{k}|^2 + m_e^2} + \sqrt{|\boldsymbol{r}|^2 + m_e^2} \\
(E_e + E_\nu)^2 &= |\boldsymbol{k}|^2 + m_e^2 + |\boldsymbol{k}|^2 + 2E_1E_2 \\
&= |\boldsymbol{k}|^2 + 2\boldsymbol{k}\cdot\boldsymbol{r} + |\boldsymbol{k}|^2 + 2E_1E_2 - 2\boldsymbol{k}\cdot\boldsymbol{r} + m_e^2 + m_\nu^2 \\
&= |\boldsymbol{k}+\boldsymbol{r}|^2 - 2k\cdot r + m_e^2 + m_\nu^2
\end{aligned}$$

So

$$\left|\frac{d}{d|\boldsymbol{q}|}(m_A - E_B - E_e - E_\nu)\right| = \frac{|\boldsymbol{q}|}{E_B} + \frac{|\boldsymbol{q}|}{E_e + E_\nu} = \frac{m_A|\boldsymbol{q}|}{E_B(E_e + E_\nu)}$$

$$\begin{aligned}
\frac{d\Gamma}{d^3\boldsymbol{k}} &= \frac{G_F^2C^2|V_{ud}|^2}{(2\pi)^4}\int_0^\pi \sin\theta d\theta\frac{|\boldsymbol{q}|^2\left[2(k\cdot q)(r\cdot q) - q^2(k\cdot r)\right]}{m_A E_B E_e E_\nu}\left(\frac{E_B(E_e+E_\nu)}{m_A|\boldsymbol{q}|}\right)|_{m_A = E_B - E_e - E_\nu} \\
&= \frac{G_F^2C^2|V_{ud}|^2}{(2\pi)^4}\int_0^\pi \sin\theta d\theta\frac{|\boldsymbol{q}|(E_e+E_\nu)\left[2(k\cdot q)(r\cdot q) - q^2(k\cdot r)\right]}{m_A^2 E_e E_\nu}
\end{aligned}$$

Conservation of momentum in the decay rest frame implies that the momenta of the products lie in a plane. We can define θ to be the angle between any two and should be able to rewrite the dot products and carry out the integral. With simplification this should result in

$$\frac{d\Gamma}{d\boldsymbol{k}} = \frac{G_F^2 C^2 |V_{ud}^2|}{2(2\pi)^4}\left[\frac{[-(k-p)^2 - m_B^2]^2}{-(k-p)^2}\right]$$

Independent of electron direction, we can integrate over $d\Omega$ and rewrite the result in terms of E_e:

$$\begin{aligned}
\frac{d\Gamma}{dE_e} &= \int \frac{d\Gamma}{d^3\boldsymbol{k}} d\Omega \cdot \frac{d|\boldsymbol{k}|}{dE_e}|\boldsymbol{k}|^2 \\
&= 4\pi \frac{d\Gamma}{d^3\boldsymbol{k}} \cdot \frac{d}{dE_e}\left(\sqrt{E_e^2 - m_e^2}\right)\cdot(E_e^2 - m_e^2) \\
&= \frac{4\pi G_F^2 C^2 |V_{ud}|^2}{2(2\pi)^4}\left[\frac{[-(k-p)^2 - m_B^2]^2}{-(k-p)^2}\right] E_e\sqrt{E_e^2 - m_e^2} \\
&= \frac{G_F^2 C^2 |V_{ud}|^2}{(2\pi)^3}\left[\frac{[m_A^2 + m_e^2 - 2E_e m_A - m_B^2]^2}{m_A^2 + m_e^2 - 2E_e m_A}\right] E_e\sqrt{E_e^2 - m_e^2} \\
&\approx \frac{G_F^2 C^2 |V_{ud}|^2}{(2\pi)^3}\left[\frac{[m_A^2 - 2E_e m_A - m_B^2]^2}{m_A^2 - 2E_e m_A}\right] E_e\sqrt{E_e^2 - m_e^2} \\
&= \frac{G_F^2 C^2 |V_{ud}|^2}{(2\pi)^3}\left(\frac{m_A E_e}{m_A - 2E_e}\right)\left[\frac{[2E_e m_A + m_B^2 - m_A^2]^2}{m_A^2}\right] E_e\sqrt{E_e^2 - m_e^2}
\end{aligned}$$

Where

$$Q^2 = m_A^2 - 2m_A m_B + m_B^2 \implies m_b^2 - m_A^2 = Q^2 - 2m_A^2 - 2m_A m_B$$

$$\begin{aligned}
\frac{d\Gamma}{dE_e} &= \frac{G_F^2 C^2 |V_{ud}|^2}{(2\pi)^3}\left(\frac{m_A E_e}{m_A - 2E_e}\right)\left[\frac{[2E_e m_A + Q^2 - 2m_A^2 - 2m_A m_B]^2}{m_A^2}\right]\sqrt{E_e^2 - m_e^2} \\
&= \frac{G_F^2 C^2 |V_{ud}|^2}{(2\pi)^3}\left(\frac{m_A E_e}{m_A - 2E_e}\right)[2E_e - 2m_A - 2m_B]^2\sqrt{E_e^2 - m_e^2}
\end{aligned}$$

$$\boxed{\frac{d\Gamma}{dE_e} = \frac{G_F^2 C^2 |V_{ud}|^2}{2\pi^3}\left(\frac{m_A E_e}{m_A - 2E_e}\right)(E_e - Q)^2\sqrt{E_e^2 - m_e^2}}$$

The minimum energy occurs when $|\boldsymbol{k}| = 0 \implies E_{min} = m_e \approx 0$. Whereas maximum energy occurs when all momentum is split amongst $\boldsymbol{k}, \boldsymbol{q}$. Neglecting m_ν so it is like a 2-body decay and neglecting m_e, $E_{\max} \approx p \approx \frac{m_A^2 - m_B^2}{2m_A} \sim Q$.

Performing the final integral,

$$\begin{aligned}
\Gamma &= \int dE_e \frac{d\Gamma}{dE_e} \\
&= \int_0^{E_{\max}} dE_e \frac{G_F^2 C^2 |V_{ud}|^2 m_A}{2\pi^3} \left(\frac{E_e}{m_A - 2E_e} \right) (E_e - Q)^2 \sqrt{E_e^2 - m_e^2} \\
&\approx \frac{G_F^2 C^2 |V_{ud}|^2 m_A}{2\pi^3} \int_0^{E_{\max}} \left(\frac{E_e^2}{m_A - 2E_e} \right) (E_e - Q)^2
\end{aligned}$$

Using Wolfram Alpha for expansion and partial fractions

$$\begin{aligned}
\Gamma \approx& \frac{G_F^2 C^2 |V_{ud}|^2 m_A}{2\pi^3} \int_0^{E_{\max}} dE_e \frac{-m_A^4 + 4m_A^3 Q - 4m_A^2 Q^2}{16(2E_e - m_A)} - \frac{1}{4} E_e^2 (m_A - 4Q) - \frac{1}{8} E_e (m_A - 2Q)^2 \\
&- \frac{1}{16} m_A (m_A - 2Q)^2 - \frac{E_e^3}{2} \\
=& \frac{G_F^2 C^2 |V_{ud}|^2 m_A}{2\pi^3} \left[\frac{-m_A^4 + 4m_A^3 Q - 4m_A^2 Q^2}{32} \ln(m_A - 2E_e) - \frac{1}{12} E_e^3 (m_A - 4Q) - \frac{1}{16} E_e^2 (m_A - 2Q)^2 \right. \\
&\left. - \frac{1}{16} E_e m_A (m_A - 2Q)^2 - \frac{E_e^4}{8} \right]_0^{E_{\max}} \\
=& \frac{G_F^2 C^2 |V_{ud}|^2 m_A^5}{2\pi^3} \left[\frac{-1 + 4\Delta - 4\Delta^2}{32} \ln(1 - 2\Delta) - \frac{1}{12} \Delta^3 (1 - 4\Delta) - \frac{1}{16} \Delta^2 (1 - 2\Delta)^2 - \frac{1}{16} \Delta (1 - 2\Delta)^2 - \frac{\Delta^4}{8} \right]_0^{E_{\max}} \\
=& \frac{G_F^2 C^2 |V_{ud}|^2 m_A^5}{2\pi^3} \left[\frac{-(2\Delta - 1)^2}{32} \ln(1 - 2\Delta) - \frac{\Delta^3}{12} + \frac{\Delta^4}{3} - \frac{\Delta^2}{16} + \frac{\Delta^3}{4} - \frac{\Delta^4}{4} - \frac{\Delta}{16} + \frac{\Delta^2}{4} - \frac{\Delta^3}{4} - \frac{\Delta^4}{8} \right] \\
=& \frac{G_F^2 C^2 |V_{ud}|^2 m_A^5}{2\pi^3} \left[\frac{-(2\Delta - 1)^2}{32} \ln(1 - 2\Delta) - \frac{\Delta^4}{24} - \frac{\Delta^3}{12} + \frac{3\Delta^2}{16} - \frac{\Delta^4}{16} \right]
\end{aligned}$$

$$\boxed{\Gamma = \frac{G_F^2 C^2 |V_{ud}|^2 m_A^5}{2\pi^3} \left(\frac{-1}{4} \right) \left[\frac{1}{2} \left(\Delta - \frac{1}{2} \right) \ln(1 - 2\Delta) + \frac{\Delta^4}{6} + \frac{\Delta^3}{3} - \frac{3\Delta^2}{4} + \frac{\Delta}{4} \right]}$$