David St-Amant Farmer
PHYS 4E03

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Assignment #6: Solutions

1 We consider $m_e \approx 0$ here and $M \equiv m_\mu$

① (4.12) gives us: $\mathcal{M} = \frac{32\pi^2\alpha^2}{t^2}\left((M^2-u^2)^2 + (s-M^2)^2 + 2tM^2\right)$

and (2.66) gives $\frac{d\sigma}{du\,dt} = \frac{-\mathcal{M}}{8\pi\,\xi(s)\,\beta(s)}\,\delta(s+t+u-2M^2)$

$$\Rightarrow \boxed{\frac{d\sigma}{du\,dt} = -\frac{4\pi\alpha^2}{\xi(s)\beta(s)}\left[\frac{(M^2-u)^2 + (s-M^2)^2 + 2tM^2}{t^2}\right]\delta\left(s+t+u-2M^2\right)}$$

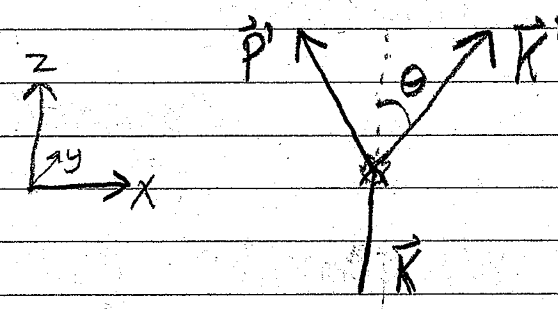
② We have $\beta \approx -4p\cdot K$ and $\xi \approx s - M^2$

Here p^μ is for the muon and K^μ for the electron. We choose in the lab frame to have the muon initially at rest, the incoming electron along z and the scattering to be in the x-z plane.

$$\Rightarrow \boxed{p^\mu = \begin{pmatrix} M \\ 0 \\ 0 \\ 0 \end{pmatrix}} \text{ and } E_e^2 = |\vec{K}|^2 + m_e^2 = |\vec{K}|^2 = W^2 \Rightarrow \boxed{K^\mu = \begin{pmatrix} W \\ 0 \\ 0 \\ W \end{pmatrix}}$$

We then have $E_\mu'^2 = |\vec{k}'|^2 = w'^2 \Rightarrow \underline{|\vec{k}'| = w'}$

In the x-z plane:



Thus, $\boxed{k'^\mu = \begin{pmatrix} w' \\ w'\sin(\theta) \\ 0 \\ w'\cos(\theta) \end{pmatrix}}$ and by conservation of $\vec{p}'$

we need $p'_x = -w'\sin(\theta)$
$p'_z = w - w'\cos(\theta)$

$$\Rightarrow \boxed{p'^\mu = \begin{pmatrix} E \\ -w'\sin(\theta) \\ 0 \\ w - w'\cos(\theta) \end{pmatrix}}$$

3) We have that $E^2 = M^2 + (w'\sin(\theta))^2 + (w - w'\cos(\theta))^2$

and $E + w' = M + w$

1) $E^2 = (M + w - w')^2 = \cancel{M^2} + \cancel{w^2} + \cancel{w'^2} + 2Mw - 2Mw' - 2ww'$

$= \cancel{M^2} + \cancel{w'^2} + \cancel{w^2} - 2ww'\cos(\theta)$

$\Rightarrow w'\left(1 + \frac{w}{M}\underbrace{\left(1 - \cos(\theta)\right)}\right) = w$

Use $\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2} \Rightarrow \underline{= 2\sin^2\left(\frac{\theta}{2}\right)}$

$$\Rightarrow \cancel{w'}\, w'\left(1+\frac{2w}{m}\sin^2(\theta/2)\right)=w$$

$$\Rightarrow \boxed{w'=\frac{w}{1+2\left(\frac{w}{m}\right)\sin^2\left(\frac{\theta}{2}\right)}}$$

2) $\boxed{t}\equiv -\left(k^\mu-k'^\mu\right)^2 = \left(w-w'\right)^2-\left(w'\sin(\theta)\right)^2$
$$-\left(w-w'\cos(\theta)\right)^2$$

$$=\cancel{w^2}+\cancel{w'^2}-2ww'-\cancel{w'^2}-\cancel{w^2}+2ww'\cos(\theta)$$

$$=-2ww'\left(1-2\cos(\theta)\right)=\boxed{-4ww'\sin^2\left(\frac{\theta}{2}\right)}$$

3) $t=-4w^2\left[\dfrac{\sin^2\left(\frac{\theta}{2}\right)}{1+\frac{2w}{m}\sin^2\left(\frac{\theta}{2}\right)}\right]$

$$\Rightarrow \boxed{dt}=\left(\frac{\partial t}{\partial\theta}\right)d\theta=(\dots)=\frac{m^2}{2}\sin(\theta)\underbrace{\left(\frac{1}{m+2w\sin^2\left(\frac{\theta}{2}\right)}\right)^2}_{=\frac{w'^2}{(mw)^2}}\left(-4w^2\right)d\theta$$

$$=\boxed{-2w'^2\sin(\theta)d\theta}$$

4) $\boxed{K'\cdot p} = -(w'M) + \vec{0} = -w'M$

5) $\boxed{K\cdot p} = -(wM) + \vec{0} = -wM$

(4) We want $\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{dt}\right)\left(\frac{dt}{\sin(\theta)\,d\theta\,d\phi}\right) = \frac{d\sigma}{dt}\left(\frac{-2w'^2\sin(\theta)\,d\theta}{2\pi\sin(\theta)\,d\theta}\right)$

$$= \frac{d\sigma}{dt}\left(\frac{-w'^2}{\pi}\right)$$

Also, $s = -(K+p)^2 = (w+M)^2 - w^2 = M^2 + 2wM$

and $u = -(K'-p)^2 = (w'-M)^2 - w'^2 = M^2 - 2Mw'$

Use $\frac{d\sigma}{du\,dt}$ from (1) with $\beta = -4p\cdot K = 4M^2w$
and $\delta = s - M^2 = 2wM$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{-4\pi\alpha^2}{(4Mw)(2wM)}\left[\frac{(2Mw')^2 + (2wM)^2 + 2M^2\left(-4ww'\sin^2\left(\frac{\theta}{2}\right)\right)}{\left(4ww'\sin^2\left(\frac{\theta}{2}\right)\right)^2}\right]\left(\frac{-w'^2}{\pi}\right)$$

$$= \frac{16\alpha^2\cancel{M^2}\cancel{w'^2}}{128\cancel{M^2}w^4\cancel{w'^2}}\left[\frac{w'^2 + w^2 - 2ww'\sin^2\left(\frac{\theta}{2}\right)}{\sin^4(\theta/2)}\right]$$

$$= \boxed{\frac{\alpha^2}{8w^2}\left[1 + \left(\frac{w'}{w}\right)^2 - 2\left(\frac{w'}{w}\right)\sin^2\left(\frac{\theta}{2}\right)\right]\csc^4\left(\frac{\theta}{2}\right)}$$

Multiply by $1 \equiv \frac{\cos^2(\theta/2)}{\cos^2(\theta/2)}$:

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4w^2} \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)} \left(\frac{w'}{w}\right) \underbrace{\left[\frac{\left(\frac{w}{w'} + \frac{w'}{w}\right)}{2\cos^2(\theta/2)} - \frac{\sin^2\left(\frac{\theta}{2}\right)}{\cos^2(\theta/2)} \right]}_{(*)}$$

$$(*) : \frac{1}{2\cos^2\left(\frac{\theta}{2}\right)} \left[\frac{w}{w'} + \frac{w'}{w} - 2\sin^2\left(\frac{\theta}{2}\right) \right] \stackrel{?}{=} 1 + \left(\frac{2ww'}{m^2}\right) \sin^2(\theta/2) \tan^2\left(\frac{\theta}{2}\right)$$

must be equal to get (4.18)

$$\Rightarrow \frac{w}{w'} + \frac{w'}{w} - 2\sin^2\left(\frac{\theta}{2}\right) \stackrel{?}{=} 2\cos^2(\theta/2) + \frac{2ww'}{m^2} \sin^4(\theta/2)$$

$$\Rightarrow \underbrace{\frac{w}{w'} + \frac{w'}{w}}_{} \stackrel{?}{=} 2 + \frac{4ww'}{m^2} \sin^4(\theta/2)$$

Use $\frac{w'}{w} = \frac{1}{1 + \frac{2w}{m}\sin^2(\theta/2)}$, so $\frac{w}{w'} = 1 + \frac{2w}{m}\sin^2(\theta/2)$

$$\Rightarrow \left(\frac{w}{w'} - 1\right)^2 = \frac{4w^2}{m^2} \sin^4(\theta/2)$$

$$\Rightarrow \frac{w^2}{w'^2} + 1 - \frac{2w}{w'} = \frac{4w^2}{m^2} \sin^4(\theta/2) \quad \Bigg\} \text{ Multiply by } \frac{w'}{w}$$

$$\Rightarrow \frac{w}{w'} + \frac{w'}{w} = 2 + \frac{4ww'}{m^2}\sin^4(\theta/2) \quad ■$$

Thus, $$\boxed{\frac{d\sigma}{d\Omega} \simeq \frac{\alpha^4}{4w^2}\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)}\left(\frac{w'}{w}\right)\left[1 + \left(\frac{2ww'}{m^2}\right)\sin^2(\theta/2)\tan^2\left(\frac{\theta}{2}\right)\right]}$$