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PHYS 4E03

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Assignment #5: Solutions

(3.4) We have that $\theta \simeq \frac{b}{mv_i^2}\int_{-\infty}^{\infty} dx \frac{F}{r}$, $r=\sqrt{x^2+b^2}$, $F=-\nabla U$

by the impulse approximation. For $U=U(r)$, we get

omit constants

① $\theta \sim \int_{-\infty}^{\infty} dx \, -\left(\frac{dU}{dr}\right)\frac{1}{r} = -\int_{-\infty}^{\infty} \frac{dx}{r}\frac{dU}{dr}$, but use chain rule since $r=r(x)$

$$\Rightarrow \frac{dU}{dr} = \frac{dU}{dx}\left(\frac{dx}{dr}\right) = \frac{dU}{dx}\left(\frac{x}{r}\right)^{-1} = \frac{r}{x}\frac{dU}{dx}$$

$$\Rightarrow \boxed{\theta \sim -\int_{-\infty}^{\infty} \frac{dx}{x}\frac{dU}{dx}}$$

② If we try and evaluate this by parts:

$$\theta \sim -\frac{U}{x}\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} U \frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{U}{x}\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx \frac{U}{x^2}$$

If we were to use that $U \to 0$ as $r \to \infty$ (so at $x \to \pm\infty$),

we would get $\theta \sim -\int_{-\infty}^{\infty} dx \frac{U}{x^2}$

However, this is incorrect because we neglected the fact there is a singularity at $x=0$ in our integrand $\frac{1}{x}\frac{dU}{dx}$ when using I.B.P.

Hence, if we then plug in $U=\frac{K}{r}$ we get an incorrect result:

$$\Rightarrow \boxed{\theta} \sim -\int_{-\infty}^{\infty} dx \frac{K}{r x^2} = -K \int_{-\infty}^{\infty} dx \frac{1}{x^2\sqrt{x^2+b^2}} \rightarrow \boxed{\infty}$$

so we get a divergent angle indicating that we indeed were not careful when using I.B.P previously.

(3.5) We again start with $\theta \simeq \frac{-b}{mv_i^2}\int_{-\infty}^{\infty} dx \frac{\nabla U}{\sqrt{x^2+b^2}}$

Here we take $U \equiv \frac{K_u}{r^n}$ for $n \geq 1$

$$\Rightarrow \theta(b) \simeq \frac{-K_u b}{mv_i^2}\int_{-\infty}^{\infty} dx \frac{1}{\sqrt{x^2+b^2}}\frac{d}{dr}\left(\frac{1}{r^n}\right) = \frac{nK_u b}{mv_i^2}\underbrace{\int_{-\infty}^{\infty} dx \frac{1}{(x^2+b^2)^{\frac{n+2}{2}}}}_{(*)}$$

After evaluating $(*)$, we will get $\frac{d\sigma}{d\Omega} = \frac{b(\theta)}{\sin(\theta)}\left|\frac{db}{d\theta}\right|$ by inverting $\theta(b) \rightarrow b(\theta)$

We need $(*) = \int_{-\infty}^{\infty} dx \, \frac{1}{(x^2+b^2)^{\frac{n+2}{2}}}$ Let $u = \frac{x}{b}$

$$= \frac{b}{b^{n+2}} \left[\int_{-\infty}^{\infty} du \, \frac{1}{(1+u^2)^{\frac{n+2}{2}}} \right] \equiv \frac{1}{b^{n+1}} \underbrace{\left[\Omega(n) \right]}_{\text{some constant depending on } n}$$

$$\Rightarrow \boxed{\theta(b) \approx \frac{n}{b^n} \frac{Ku}{m v_i^2} \Omega(n), \quad \underset{[\text{Using Mathematica}]}{\Omega(n)} = \sqrt{\pi} \, \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}+1\right)}}$$

Thus, $b(\theta) = \left[\frac{1}{\theta} \left(\frac{n \Omega(n) Ku}{m v_i^2} \right) \right]^{1/n}$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{b(\theta)}{\sin(\theta)} \left| \frac{db}{d\theta} \right| = \frac{b(\theta)}{\sin(\theta)} \left| \left(\frac{-1}{n} \right) \frac{b(\theta)}{\theta} \right|$$

$$= \boxed{\frac{1}{n \theta^{\left(\frac{2}{n}+1\right)} \sin(\theta)} \left(\frac{n \Omega(n) Ku}{m v_i^2} \right)^{2/n}}$$

(3.9) The Born approximation gives us that

$$\frac{d\sigma}{d\Omega} \simeq \frac{m^2}{4\pi^2} \underbrace{\left| \int d^3x\, U(\vec{x}) e^{-i\vec{q}\cdot\vec{x}} \right|}_{(*)}$$, where $\vec{q} \equiv \vec{k}_f - \vec{k}_i$ is the momentum transfer

Here we consider $U(\vec{r}) = K\delta^3(\vec{r})$

Since $U(\vec{r})$ depends only on $|\vec{r}|$ and $\delta^3(\vec{r}) = \frac{\delta(r)}{4\pi r^2}$

we get from (3.125):

$$\Rightarrow \frac{d\sigma}{d\Omega} \simeq \frac{4m^2K^2}{q^2}\left[\int_0^\infty dr\, \frac{r\delta(r)\sin(qr)}{4\pi r^2}\right]^2$$

$$= \frac{m^2K^2}{4\pi^2 q^2}\left[\underbrace{\left.\frac{\sin(qr)}{r}\right|_{r=0}}_{\to q}\right]^2$$

$$= \frac{m^2K^2}{4\pi^2}$$