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PHYS 4E03

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Assignment #4: Solutions

1 We consider $A + B \to C + D$ with $m_A = m_C$, $m_B = m_D$

Let in Lab frame: $\vec{p}_B = 0$, $\vec{p}_A = p\hat{z}$

① $s = E_{cm}^2 = -2p_A \cdot p_B + m_A^2 + m_B^2$ } Lab frame

$$= -2\left(-E_A^{lab}\underbrace{E_B}_{=m_B} + \cancel{\vec{p}_A \cdot \vec{p}_B}^{\,0}\right) + m_A^2 + m_B^2$$

$$\Rightarrow \boxed{E_{cm} = \sqrt{2E_A^{lab} m_B + m_A^2 + m_B^2}}$$

② In COM frame, $|\vec{p}_A^{\,com}| = |\vec{p}_B^{\,com}|$: [Let $E_{()}^{cm} \equiv E_{()}$ for simplicity]

$$|\vec{p}_A^{\,com}|^2 = |\vec{p}_B^{\,com}|^2 \Rightarrow E_A^2 - m_A^2 = E_B^2 - m_B^2 \quad \text{, use } E_{cm} = E_A + E_B$$

$$\Rightarrow \cancel{E_A^2} - m_A^2 = (E_{cm} - E_A)^2 - m_B^2 = E_{cm}^2 - 2E_{cm}E_A + \cancel{E_A^2} - m_B^2$$

$$\Rightarrow \boxed{E_A = \frac{E_{cm}^2 + m_A^2 - m_B^2}{2E_{cm}}}, \quad \boxed{E_B = \frac{E_{cm}^2 + m_B^2 - m_A^2}{2E_{cm}}}$$

③ We can apply the same derivation using $|\vec{p}_C| = |\vec{p}_D|$:

$$\Rightarrow \boxed{E_C} = \frac{E_{cm}^2 + m_C^2 - m_D^2}{2E_{cm}} = \boxed{\frac{\left(2E_A^{lab} m_B + m_A^2 + m_B^2\right) + m_C^2 - m_D^2}{2\sqrt{2E_A^{lab} m_B + m_A^2 + m_B^2}}}$$

For $m_A = m_C$ and $m_B = m_D$:

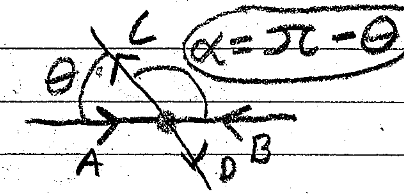
$$\Rightarrow \boxed{E_C = \frac{E_A^{lab}\, m_B + m_A^2}{\sqrt{2E_A^{lab}\, m_B + m_A^2 + m_B^2}}}$$

$$\Rightarrow \boxed{E_D} = \frac{E_{cm}^2 + m_D^2 - m_C^2}{2E_{cm}} = \boxed{\frac{E_A\, m_B + m_B^2}{\sqrt{2E_A^{lab}\, m_B + m_A^2 + m_B^2}}}$$

(4) $\boxed{u} \equiv -(p_A - p_D)^2$, but $p_A + p_B = p_C + p_D$

$$= -\left(\cancel{p_A} - (\cancel{p_A} + p_B - p_C)\right)^2 = \boxed{-(p_B - p_C)^2}$$

(5) Want $E_C^{lab} = f(E_A^{lab}, \theta)$ In COM:



Lab: $\boxed{u} = -(p_B - p_C)^2 = 2p_B \cdot p_C + m_B^2 + m_C^2 = \boxed{-2E_C^{lab}\, m_B + m_B^2 + m_C^2}$ } (A)

COM: $\boxed{u} = 2p_B \cdot p_C + m_B^2 + m_C^2 = 2\left(-E_B E_C + |\vec{p}_B^{cm}||\vec{p}_C^{cm}|\cos(\alpha)\right) + m_B^2 + m_C^2$ → see above sketch

$$= -2E_B E_C - 2|\vec{p}_B^{cm}||\vec{p}_C^{cm}|\cos(\theta) + m_B^2 + m_C^2$$ } (B)

⇒ Take (A) = (B) and solve for E_C^{lab}:

$$\boxed{E_C^{lab} = \frac{E_B E_C + |\vec{p}_B^{cm}||\vec{p}_C^{cm}|\cos(\theta)}{m_B}}$$, where E_B, E_C are from (3)

$\boxed{2}$ We consider the same scattering as in $\boxed{1}$,

$A + B \to C + D$ with $m_A = m_C$, $m_B = m_D$

① We want $\left(\frac{d\sigma}{dE_c}\right)^{lab} = \left(\frac{d\sigma}{d\Omega}\right)^{Lab} \left(\frac{d\Omega}{dE_c}\right)^{Lab}$

$$\Rightarrow \left(\frac{d\Omega}{dE_c}\right)^{Lab} = \left(\frac{dE_c^{lab}}{2\pi\sin(\theta)d\theta}\right)^{-1} = 2\pi\sin(\theta)\underbrace{\left(\frac{dE_c^{lab}}{d\theta}\right)^{-1}}_{=\left(-|\vec{p}_B^{\,com}||\vec{p}_c^{\,com}|\frac{\sin(\theta)}{m_b}\right)^{-1}}$$

From (2.51): $\left(\frac{d\sigma}{d\Omega}\right)^{Lab} = \frac{\mathcal{M}|\vec{p}_c|^2}{(8\pi)^2 m_B|\vec{p}_A|\,|E_D^{Lab}|\vec{p}_c| - E_c^{Lab}|\vec{p}_A|\cos(\theta)|}$

and from (2.53): $\left(\frac{d\sigma}{d\Omega}\right)^{com} = \frac{\mathcal{M}|\vec{p}_c^{\,com}|}{(8\pi)^2|\vec{p}_A^{\,com}|\underbrace{(E_A+E_B)^2}_{E_{cm}}} \equiv l^2$ here since we suppose an isotropic diff cross-section

$$\Rightarrow \mathcal{M} = (8\pi)^2 l^2 E_{cm}^2 \frac{|\vec{p}_A^{\,com}|}{|\vec{p}_c^{\,com}|}$$

Thus, $\boxed{\left(\frac{d\sigma}{d\Omega}\right)^{Lab}} = (8\pi)^2 l^2 E_{cm}^2 \frac{|\vec{p}_A^{\,com}|}{|\vec{p}_c^{\,com}|}\frac{|\vec{p}_c|^2}{(8\pi)^2}\frac{1}{m_b|\vec{p}_A|\,|E_D^{lab}|\vec{p}_c| - E_c^{lab}|\vec{p}_A|\cos(\theta)|}$

$$\boxed{= \frac{l^2}{m_b}\frac{|\vec{p}_c|^2}{|\vec{p}_A|}\frac{|\vec{p}_A^{\,com}|}{|\vec{p}_c^{\,com}|}\frac{E_{cm}^2}{|E_D^{lab}|\vec{p}_c| - E_c^{lab}|\vec{p}_A|\cos(\theta)|}}$$

$$\Rightarrow \boxed{\left(\frac{d\sigma}{dE_c}\right)^{lab}} = 2\pi \cancel{\sin(\theta)} \left(\frac{\cancel{m6}}{\cancel{\sin(\theta)}\, \cancel{|\vec{p}_B^{\,com}|}\, |\vec{p}_c^{\,com}|}\right) \frac{l^2}{\cancel{m6}} \frac{|\vec{p}_c|^2}{|\vec{p}_A|} \frac{\cancel{|\vec{p}_A^{\,com}|}}{|\vec{p}_c^{\,com}|} \frac{E_{cm}^2}{\left|E_D^{lab}|\vec{p}_c| - E_c^{lab}|\vec{p}_A| \cos(\theta)\right|}$$

$|\vec{p}_A^{\,com}| = |\vec{p}_B^{\,com}|$

$|\vec{p}_A| \equiv |p\Sigma| = p$

$$= \boxed{\frac{2\pi l^2}{p} \frac{|\vec{p}_c|^2}{|\vec{p}_c^{\,com}|^2} \frac{E_{cm}^2}{\left|E_D^{lab}|\vec{p}_c| - E_c^{lab}\, p \cos(\theta)\right|}}$$

with $|\vec{p}_c|^2 = (E_c^{lab})^2 - m_c^2$, $|p_c^{com}|^2 = E_c^2 - m_c^2$

and E_{cm}^2, E_c^{lab}, E_D^{lab} and E_D^{lab} obtained in $\boxed{1}$

$\boxed{3}$ We want $\theta(b)$ for two particles of relative speed v_i, and interaction potential $U(r) = \frac{K}{r^2} > 0$

① $\theta(b) = \pi - 2\phi_0(b)$, where we get $\phi_0(b)$ using

$$\phi_0 = \int_{r_0}^{\infty} \frac{dr}{(dr/d\phi)}, \quad \underbrace{\frac{b^2}{r_0^2} = 1 - \frac{2U(r_0)}{mv_i^2}}_{\text{From (3.12)}} = 1 - \frac{2K}{mv_i^2} \cdot \frac{1}{r_0^2}$$

$$\Rightarrow r_0^2 = b^2 + \frac{2K}{mv_i^2} \Big\} (*)$$

Also, $\frac{dr}{d\phi} = \frac{1}{b} r \sqrt{r^2 - b^2 - \frac{2r^2 U(r)}{mv_i^2}}$ $\Big\}$ From (3.13)

$$= \frac{1}{b} r \sqrt{r^2 - b^2 - \frac{2K}{mv_i^2}}$$

$$= \frac{r}{b}\sqrt{r^2 - r_0^2} \quad \text{using } (*)$$

Thus, $\boxed{\phi_0(b)} = b \int_{r_0}^{\infty} \frac{dr}{r\sqrt{r^2 - r_0^2}} = \boxed{\frac{b\pi}{2r_0}}$ [using Mathematica]

$$\Rightarrow \theta(b) = \pi - 2\left(\frac{b\pi}{2r_0}\right) = \pi\left(1 - \frac{b}{r_0}\right)$$

$$\Rightarrow \boxed{b(\theta) = r_0\left(1 - \frac{\theta}{\pi}\right)}$$

Plugging in $\theta()$:

$$b^2 = \left(b^2 + \frac{2K}{mv_i^2}\right)\left(1 - \frac{\theta}{\pi}\right)^2 \Rightarrow \boxed{b = \sqrt{\frac{2K}{mv_i^2}}\frac{(\pi - \theta)}{\sqrt{(2\pi - \theta)\theta}}}$$

(2) We want $\frac{d\sigma}{d\Omega} = \frac{b}{\sin(\theta)}\left|\frac{db}{d\theta}\right|$ } From (3.15)

$$\Rightarrow \left|\frac{db}{d\theta}\right| = \sqrt{\frac{2K}{mv_i^2}}\frac{\pi^2}{[(2\pi - \theta)\theta]^{3/2}}$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{2K}{mv_i^2}\frac{\pi^2(\pi - \theta)}{\sin(\theta)((2\pi - \theta)\theta)^2}}$$

(3) For $\theta \approx 0$, $\sin(\theta) \approx \theta$

$$\boxed{\frac{d\sigma}{d\Omega}} \rightarrow \frac{2K}{mv_i^2}\frac{\pi^3}{\theta^3} \boxed{\propto \theta^{-3}}$$

Whereas for Rutheford scattering with $U = \frac{K}{r}$ we had $\frac{d\sigma}{d\Omega} \propto \theta^{-4}$

$\Rightarrow$ Rutherford is stronger since $U(r)$ fall offs less rapidely over r.

4 We have the time-dependent Schrödinger equation

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi \quad \} (*)$$

Also, $\rho = \psi^*\psi$ and $\vec{j} = \frac{i}{2m}\left(\psi\vec{\nabla}\psi^* - \psi^*\vec{\nabla}\psi\right)$

① Take $\vec{\nabla}\cdot\vec{j}$ and use the vector identity for the divergence: $\vec{\nabla}\cdot(g\vec{A}) = g\vec{\nabla}\cdot\vec{A} + (\vec{\nabla}g)\cdot\vec{A}$

$$\Rightarrow \boxed{\vec{\nabla}\cdot\vec{j}} = \frac{i}{2m}\left[\left(\psi\nabla^2\psi^* + \cancel{\vec{\nabla}\psi\cdot\vec{\nabla}\psi^*}\right) - \left(\psi^*\nabla^2\psi + \cancel{\vec{\nabla}\psi^*\cdot\vec{\nabla}\psi}\right)\right]$$

$$= \frac{i}{2m}\left[\psi\left(-2mi\frac{\partial\psi}{\partial t}\right)^* - \psi^*\left(-2mi\frac{\partial\psi}{\partial t}\right)\right]$$

(Where we use $(*)$ to sub in $\nabla^2\psi$, $\nabla^2\psi^*$)

$$= -\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t} = -\frac{\partial}{\partial t}(\psi\psi^*) = \boxed{-\frac{\partial\rho}{\partial t}}$$

Thus, $\boxed{\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0}$