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PHYS 4E03

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Assignment #3: Solutions

$\boxed{1.5}$ Normal distribution: $\mathcal{N}(\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$

(A) $\mu_A = 270$, $\sigma_A = 15$ [Using Mathematica]

- $N_A(t<240) = 1000 \cdot \left(\int_{-\infty}^{240} \mathcal{N}(\mu_A, \sigma_A^2)\, dt \right)$

$\approx \boxed{23}$

- $N_A(t<225) = 1000 \cdot \left(\int_{-\infty}^{225} \mathcal{N}(\mu_A, \sigma_A^2)\, dt \right)$

$\approx \boxed{1}$

(B) $\mu_B = 270$, $\sigma_B = 20$

- $N_B(t<240) = (\dots) = \boxed{67}$
- $N_B(t<225) = (\dots) = \boxed{12}$

1.9 Poisson distribution: $\mu = N$, so $\sigma = \sqrt{\mu} = \sqrt{N}$

$$P(n,\mu) := \frac{\mu^n e^{-\mu}}{n!}$$

We want the probability to find $n > N + 3\sigma$,

so $n > N + 3\sqrt{N}$.

$$\Rightarrow \text{Prob}(n > N + 3\sqrt{N}) = \sum_{n=N+3\sqrt{N}}^{\infty} P(n,N)$$

$$= \sum_{n=N+3\sqrt{N}}^{\infty} \frac{N^n e^{-N}}{n!}$$

Or, $= 1 - \text{Prob}(n < N + 3\sqrt{N}) = \boxed{1 - \sum_{n=0}^{N+3\sqrt{N}} \frac{N^n e^{-N}}{n!}}$

To find $n > N + 5\sigma$, we have:

$$\Rightarrow \text{Prob}(n > N + 5\sqrt{N}) = 1 - \text{Prob}(n < N + 5\sqrt{N})$$

$$= \boxed{1 - \sum_{n=0}^{N+5\sqrt{N}} \frac{N^n e^{-N}}{n!}}$$

There is no closed form for these expressions, so for a given N, we need to evaluate this <u>numerically</u>.

2.9 We have $\mathcal{M}(e^+e^- \to \mu^+\mu^-) = \frac{32\pi^2\alpha^2}{s^2}(u^2+t^2)$

① $$\frac{d\sigma}{du\,dt} := \frac{-\mathcal{M}}{8\pi\xi(s)f(s)}\,\delta(s+t+u - m_{e^+}^2 - m_{e^-}^2 - m_{\mu^+}^2 - m_{\mu^-}^2)$$

where $f(s) := 2\sqrt{(s-(m_{e^+}+m_{e^-})^2)(s-(m_{e^+}-m_{e^-})^2)}$

$$\xi(s) := \sqrt{(s-m_{e^+}^2-m_{e^-}^2)^2 - 4m_{e^+}^2 m_{e^-}^2} = \frac{f(s)}{2}$$

We are in the ultra-relativistic regime, so

$s,t,u \gg m_{e^+}, m_{e^-}, m_{\mu^+}, m_{\mu^-}$

$$\Rightarrow f(s) \to 2s,\quad \xi(s) \to s,\quad \delta((\dots)) \to \delta(s+t+u)$$

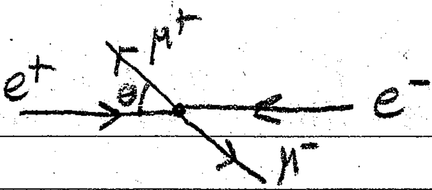
$$\Rightarrow \frac{d\sigma}{du\,dt} = \frac{-\mathcal{M}}{8\pi(s)(2s)}\,\delta(s+t+u) = \boxed{\frac{-\mathcal{M}}{16\pi s^2}\,\delta(s+t+u)}$$

② $$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega}, \qquad d\Omega = \sin(\theta)\,d\theta\,d\phi$$

$$= 2\pi\sin(\theta)\,d\theta$$ since nothing depends on ϕ in the COM frame

$$= \left(\frac{d\sigma}{dt}\right)\left(\frac{dt}{d\theta}\right)\left(\frac{1}{2\pi\sin(\theta)}\right)$$

In the COM frame,



$$t := 2\, p_{e^+} \cdot p_{\mu^+} + m_{e^+}^2 + m_{\mu^+}^2$$

$$= 2\left(-E_{e^+} E_{\mu} + \vec{p}_{e^+} \cdot \vec{p}_{\mu}\right) + m_{e^+}^2 + m_{\mu^+}^2$$

$$= 2\left(-E_{e^+} E_{\mu^+} + |p_{e^+}||p_{\mu^+}| \cos(\theta)\right) + m_{e^+}^2 + m_{\mu^+}^2$$

where $|p_{e^+}| = \sqrt{E_{e^+}^2 - m_{e^+}^2}$

In the ultra-relativistic regime, $p_{e^+} \gg m_{e^+}$, etc.

so $E_{e^+} \approx |p_{e^+}|$, etc.

$$\Rightarrow t \approx 2\left(-E_{e^+} E_{\mu^+} + E_{e^+} E_{\mu^+} \cos(\theta)\right)$$

$$= -2 E_{e^+} E_{\mu^+} \left(1 - \cos(\theta)\right) = -4 E_{e^+} E_{\mu^+} \sin^2\left(\frac{\theta}{2}\right)$$

Also in the COM frame, $E_{e^+} \approx \frac{E_{cm}}{2}$, $E_{\mu^-} \approx \frac{E_{cm}}{2}$

$$\Rightarrow \underline{\underline{t}} \approx -E_{cm}^2 \sin^2\left(\frac{\theta}{2}\right) = \underline{\underline{-s \sin^2\left(\frac{\theta}{2}\right)}}$$

Thus, $$\underline{\underline{\frac{dt}{d\theta}}} = -\left(2 E_{e^+} E_{\mu^+}\right) \sin(\theta) = \frac{-E_{cm}^2}{2} \sin(\theta) = \underline{\underline{\frac{-s}{2} \sin(\theta)}}$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{dt}\right)\left(\frac{-s}{2}\sin(\theta)\right)\left(\frac{1}{2\pi\sin(\theta)}\right)$$

$$= \frac{d\sigma}{dt}\left(\frac{-s}{4\pi}\right)$$

$$\Rightarrow \underline{\frac{d\sigma}{dt}} = \int\left(\frac{d\sigma}{dt\,du}\right)du = \int \frac{-\left(\frac{32\pi^2\alpha^2}{s^2}\right)}{16\pi s^2}\left(u^2+t^2\right)\delta(s+t+u)\,du$$

$$= \frac{-2\pi\alpha^2}{s^4}\left((s+t)^2+t^2\right), \text{ but } \underset{\text{before}}{t = -s\sin^2\left(\tfrac{\theta}{2}\right)} \text{ from}$$

$$\left.\begin{aligned} &= \frac{-2\pi\alpha^2}{s^2}\left(\left(1-\sin^2\left(\tfrac{\theta}{2}\right)\right)^2+\sin^4\left(\tfrac{\theta}{2}\right)\right) \\ &= \frac{-2\pi\alpha^2}{s^2}\left(\cos^4\left(\tfrac{\theta}{2}\right)+\sin^4\left(\tfrac{\theta}{2}\right)\right)\end{aligned}\right\} \text{use trigo. identities}$$

$$\underline{= \frac{-\pi\alpha^2}{s^2}\left(1+\cos^2(\theta)\right)}$$

Thus, $\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}\left(1+\cos^2(\theta)\right)}$

③ $\boxed{\sigma} = \int \frac{d\sigma}{d\Omega}\, d\Omega = \int \frac{\alpha^2}{4s}\left(1+\cos^2(\theta)\right)\sin(\theta)\, d\theta\, d\phi$

$$= \frac{\pi\alpha^2}{2s} \underbrace{\int_0^{\pi} \left(1+\cos^2(\theta)\right)\sin(\theta)\, d\theta}_{=\frac{8}{3} \text{ using Mathematica}}$$

$$= \boxed{\frac{4\pi\alpha^2}{3s}}$$

④ For $E_{cm} = 10\ \text{GeV}$, so $s = 100\ \text{GeV}^2$, $\alpha = \frac{1}{137}$

$\Rightarrow \boxed{\sigma} = 2.23 \times 10^{-6}\,(\text{GeV})^{-2}$, $\quad 1\,(\text{GeV})^{-2} = 0.3894\ \text{mb}$

$= \boxed{0.87\ \text{nb}}$

$\boxed{2.11}$ We have $\mathcal{M} = \frac{(4\pi\alpha_Z)^2}{|s-m^2-im\Gamma|^2}\left[(g_L^4+g_R^4)u^2 + 2g_L^2g_R^2t^2\right]$

① $\frac{d\sigma}{du\,dt} := \frac{-\mathcal{M}}{8\pi S(s)\beta(s)}\,\delta(s+t+u-m_{e^+}^2-m_{e^-}^2-m_{\mu^+}^2-m_{\mu^-}^2)$

Just like in $\boxed{2.9}$, we consider the ultra-relativistic regime, so we get the same result, but with our new $\mathcal{M}$:

$$\Rightarrow \boxed{\frac{d\sigma}{du\,dt} = \frac{-\mathcal{M}}{16\pi s^2}\,\delta(s+t+u)}$$

② In the COM frame, we will also get the same expression as in $\boxed{2.9}$ for $\frac{d\sigma}{d\Omega}$:

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{-s}{4\pi}\left(\frac{d\sigma}{dt}\right)$$

However, here $\frac{d\sigma}{dt}$ is different than before:

$$\Rightarrow \frac{d\sigma}{dt} = \int\left(\frac{d\sigma}{du\,dt}\right)du = \frac{-1}{16\pi s^2}\underbrace{\int \mathcal{M}\,\delta(s+t+u)\,du}_{(*)}$$

$$(*): \frac{(4\pi\alpha_Z)^2}{|s-M^2-iM\Gamma|^2}\int \underbrace{\left[(g_L^4+g_R^4)u^2+2g_L^2g_R^2t^2\right]}_{(+)}\delta(s+u+t)\,du$$

$$(+) = (g_L^4+g_R^4)(s+t)^2 + 2g_L^2g_R^2t^2$$

$$\Rightarrow \boxed{\frac{d\sigma}{d\Omega}} = \left(\frac{-s}{4\pi}\right)\left(\frac{-1}{16\pi s^2}\right)\left(\frac{(4\pi\alpha_Z)^2}{|s-M^2-iM\Gamma|^2}\right)\left[(g_L^4+g_R^4)(s+t)^2+2g_L^2g_R^2t^2\right]$$

$$= \boxed{\frac{\alpha_Z^2}{4s|s-M^2-iM\Gamma|^2}\left((g_L^4+g_R^4)(s+t)^2+2g_L^2g_R^2t^2\right)}$$

Recall we had in $\boxed{2.9}$ in the COM frame:

$$t = -s\sin^2\left(\frac{\theta}{2}\right) = -\frac{s}{2}\left(1-\cos(\theta)\right), \quad dt = -\frac{s}{2}\sin(\theta)\,d\theta$$

$$\Rightarrow (g_L^4+g_R^4)(s+t)^2 + 2g_L^2g_R^2t^2 = s^2\left[(g_L^4+g_R^4)\cos^4\left(\frac{\theta}{2}\right)+2g_L^2g_R^2\sin^4\left(\frac{\theta}{2}\right)\right]$$

Also, we have in the denominator of $\frac{d\sigma}{d\Omega}$ the factor $(s-M^2)^2+(M\Gamma)^2 = (E_{cm}^2-M^2)^2+(M\Gamma)^2$ which is a Breit-Wigner factor with the Ecm-dependence.

(3) We can now get the cross section:

$$\Rightarrow \sigma = \int \left(\frac{d\sigma}{d\Omega}\right) d\Omega = \int \frac{d\sigma}{d\Omega} \sin(\theta)\, d\theta\, d\phi = 2\pi \int \left(\frac{d\sigma}{d\Omega}\right) \sin(\theta)\, d\theta$$

$$= \frac{\pi \alpha_Z^2 s}{2\left((s-M^2)^2 + (M\Gamma)^2\right)} \underbrace{\int \left(\left[g_L^4 + g_R^4\right] \cos^4\left(\tfrac{\theta}{2}\right) + 2 g_L^2 g_R^2 \sin^4\left(\tfrac{\theta}{2}\right)\right) \sin(\theta)\, d\theta}_{= \frac{2}{3}\left(g_L^2 + g_R^2\right)^2 \text{ using Mathematica}}$$

$$\Rightarrow \boxed{\sigma = \frac{\pi \alpha_Z^2 s \left(g_L^2 + g_R^2\right)^2}{3\left((s-M^2)^2 + (M\Gamma)^2\right)}}$$

(4) For $M \approx 90$ GeV, $\Gamma = 2.4$ GeV, $s_W^2 = 0.23$, $s \approx 90^2\,\text{GeV}^2$

we have $\alpha_Z \approx 0.0412$, $g_L = -0.27$, $g_R = 0.23$

$$\Rightarrow \boxed{\sigma \approx 0.49\,(\text{GeV})^{-2} \approx 1.9\text{ nb}}$$

$\boxed{2.12}$ We want $\sigma_{prompt} := \sigma_p$ from $\boxed{2.9}$ and $\sigma_{resonant} := \sigma_r$ from $\boxed{2.11}$ for $E = M = 90\,GeV$:

① $\sigma_p = \dfrac{4\pi\alpha^2}{3(90\,GeV)^2} \approx 0.0107\,nb$

② $\sigma_r \approx 1.9\,nb$

For $\mathcal{L} = 10^{32}\dfrac{1}{s\cdot cm^2}$, the event rate given by

$\Delta N := \sigma \mathcal{L}$ is then:

$$\boxed{\Delta N_p} = \sigma_p \cdot \mathcal{L} = 0.0107\,nb \cdot \underbrace{\left(10^{32}\frac{1}{s\cdot cm^2}\cdot\frac{1b}{10^9\,nb}\cdot\frac{10^{-24}\,cm^2}{1b}\right)}_{=0.1\ s^{-1}nb^{-1}}$$

$$\approx \boxed{1.07\times10^{-3}\ s^{-1}}$$

$$\boxed{\Delta N_r} = \sigma_r \cdot \mathcal{L} \approx \boxed{0.19\ s^{-1}}$$

(?) ③ We want the time τ required for

$$\Delta N_r \tau > \Delta N_p \tau + 3\sqrt{\Delta N_p \tau} \Rightarrow \boxed{\tau = 0.27\,s}$$

④ For 5σ, $\tau = 0.75\,s$

$$\Rightarrow \boxed{L} = \mathcal{L}\cdot\tau = (0.1\ s^{-1}\cdot cm^{-2})\cdot(0.75\,s) = 0.075\ cm^{-2}$$