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(1)

$$(p')^\mu = \Lambda_z p^\nu$$

$$= \begin{pmatrix} ch_z & \cdot & \cdot & sh_z \\ \cdot & 1 & \cdot & \cdot \\ \cdot & \cdot & 1 & \cdot \\ sh_z & \cdot & \cdot & ch_z \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

$$= \begin{pmatrix} ch_z E + sh_z p_z \\ p_x \\ p_y \\ sh_z E + ch_z p_z \end{pmatrix}.$$

Thus,
$$E' = ch_z E + sh_z p_z$$
$$p_x' = p_x$$
$$p_y' = p_y$$
$$p_z' = sh_z E + ch_z p_z$$

It is immediately apparent that $dp_x$ and $dp_y$ are invariant since

$$dp_x' = dp_x \quad , \quad dp_y' = dp_y .$$

Showing that $dp_z$ is invariant requires a little more maneuvering.

① cont'd

We note that

$$\frac{dp_z'}{dp_z} = \frac{dE}{dp_z} sh_z + ch_z ,$$

$$\cancel{\frac{dE'}{dE} = ch_z + \frac{dp_z}{E} sh_z}$$ don't need

$$\frac{dE}{dp_z} = \frac{d}{dp_z}\left[\vec{p}\cdot\vec{p} + m^2\right]^{1/2}$$

$$= \frac{d}{dp_z}\left[p_x^2 + p_y^2 + p_z^2 + m^2\right]^{1/2}$$

$$= p_z\left[p_x^2 + p_y^2 + p_z^2 + m^2\right]^{-1/2} = \frac{p_z}{E}.$$

Therefore,

$$\frac{dp_z'}{dp_z} = \frac{p_z}{E} sh_z + ch_z = \frac{E'}{E}$$

$\Rightarrow \frac{dp_z'}{E'} = \frac{dp_z}{E}$ , showing that $\frac{dp}{E}$ too is invariant.

(2) The two-body differential decay rate is given by

$$d\Gamma(A\to BC) = \frac{1}{2E_A}\mathcal{M}(2\pi)^4\,\delta^4(p_A^\mu - p_B^\mu - p_C^\mu)\,\frac{d^3p_B\,d^3p_C}{(2\pi)^6\,4E_BE_C}$$

$$= \frac{\mathcal{M}}{8E_AE_BE_C(2\pi)^2}\,\delta(E_A - E_B - E_C)\,\delta^3(p_A - p_B - p_C)\,d^3p_B\,d^3p_C$$

Since we are in the rest frame of A then $p_A = 0$ and $E = \sqrt{p^2 + m_A^2} = m_A$.

$$d\Gamma = \frac{\mathcal{M}}{8m_AE_BE_C(2\pi)^2}\,\delta(m_A - E_B - E_C)\,\delta^3(p_C + p_B)\,d^3p_B\,d^3p_C$$

Integrating w.r.t. $p_C$, $\delta^3(p_C + p_B)$ imposes that $p_B = -p_C$,

$$d\Gamma = \frac{\mathcal{M}}{8m_AE_BE_C(2\pi)^2}\,\delta(m_A - E_B - E_C)\,d^3p_B\,\Big|_{p_B=-p_C}$$

When considering $d^3p_B$ we turn to spherical coordinates

$$d^3p_B = p^2\,dp\,d\Omega$$

where we drop the subscript B since $p^2 = p_B^2 = p_C^2$.

$$d\Gamma = \frac{\mathcal{M}}{8m_AE_BE_C(2\pi)^2}\,\delta(m_A - E_B - E_C)\,p^2\,dp\,d\Omega\,\Big|_{p_B=-p_C}$$

② cont'd

To handle the Delta function we note

$$\int dy\, \delta[g(x,y)]\, f(y) = \int \frac{dg}{|\partial g/\partial y|} \delta(g) f(y) = \left( \frac{f(y)}{|\partial g/\partial y|} \right)_{y=y(x)}$$

where $y = y(x)$ is the sol'n to $g(x,y) = 0$.

With $g(m_A, p) = m_A - E_{tot} = m_A - \sqrt{p^2+m_B^2} - \sqrt{p^2+m_C^2}$ ; $E_{tot} = E_B + E_C$

$$\frac{\partial g}{\partial p} = -p\left[\frac{1}{E_B} + \frac{1}{E_C}\right] = -p\frac{(E_B+E_C)}{E_B E_C} = \frac{-pE_{tot}}{E_B E_C}$$

Thus,

$$d\Gamma = \frac{M}{8 m_A E_B E_C (2\pi)^2} \left(\frac{E_B E_C}{p E_{tot}}\right) p^2\, \delta(m_A - E_{tot})\, dE_{tot}\, d\Omega \Big|_{p_B = -p_C}$$

$$\boxed{\frac{d\Gamma}{d\Omega} = \frac{M p}{32\pi^2 m_A^2}}$$

$E_{tot} \to m_A$ from $\delta()$

Given that $E_B = \dfrac{m_A^2 + m_B^2 - m_C^2}{2m_A}$

$E_C = \dfrac{m_A^2 + m_C^2 - m_B^2}{2m_A}$

$$p = \sqrt{E_B^2 - m_B^2} = \frac{\sqrt{(m_A^2 + m_B^2 - m_C^2)^2 - 4m_A^2 m_B^2}}{2m_A}$$

$$= (m_A^4 + m_B^4 + m_C^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 - 2m_B^2 m_C^2)^{1/2} / 2m_A$$

Considering the argument under the sqrt we are trying to prove:

$$[m_A^2 - (m_B + m_C)^2][m_A^2 - (m_B - m_C)^2]$$

Nice!

$$= m_A^4 - m_A^2[(m_B + m_C)^2 + (m_B - m_C)^2] + (m_B + m_C)^2 (m_B - m_C)^2$$

$$= m_A^4 - m_A^2[2m_B^2 + 2m_C^2] + (m_B^2 - m_C^2)^2$$

$$= m_A^4 - 2m_A^2 m_B^2 - 2m_A^2 m_C^2 + m_B^4 + m_C^4 - 2m_B^2 m_C^2$$

which is what we found above.

③ Using the result of (2):

$$\frac{d\Gamma}{d\Omega} = p^2 \frac{(G_F |V_{ud}| m_\mu F_\pi)^2}{4\pi^2 m_\pi^2}$$

Since the neutrino mass $m_\nu \ll m_\mu$,

$$p \simeq \left[\frac{(m_\pi^2 - m_\mu^2)(m_\pi^2 - m_\mu^2)}{2m_\pi}\right]^{1/2}$$

$$p^2 = \frac{(m_\pi^2 - m_\mu^2)^2}{4m_\pi^2} = \frac{1}{4m_\pi^2}\left[m_\pi^4 + m_\mu^4 - 2m_\pi^2 m_\mu^2\right]$$

$$= \frac{m_\pi^2}{4}\left[1 - 2\frac{m_\mu^2}{m_\pi^2} + \frac{m_\mu^4}{m_\pi^2}\right]$$

$$= \frac{m_\pi^2}{4}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2$$

$$d\Gamma = \frac{(G_F |V_{ud}| m_\mu F_\pi)^2 m_\pi}{16\pi^2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 d\Omega$$

Since none of the elements have $\theta, \phi$ dependencies,

$$\Gamma = \frac{(G_F |V_{ud}| m_\mu F_\pi)^2 m_\pi}{16\pi^2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2 (4\pi)$$

$$\boxed{\Gamma = \frac{G_F^2 |V_{ud}|^2 F_\pi^2 m_\mu^2 m_\pi}{4\pi}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^2}$$

③ cont'd

For the decay to $e^+\nu_e$, we change $m_\mu \rightarrow m_e = 0.511\,\text{MeV}$

$$R_\pi = \frac{\Gamma(\pi^+ \rightarrow e^+\nu_e)}{\Gamma(\pi^+ \rightarrow \mu^+\nu_\mu)}$$

$$= \frac{m_e^2}{m_\mu^2}\left(1 - \frac{m_\mu^2}{m_\pi^2}\right)^{-2}\left(1 - \frac{m_e^2}{m_\pi^2}\right)^{2}$$

$$= 1.237 \times 10^{-4}$$

④ Beginning with

$$d\Gamma = \frac{\mathcal{M}\, p\, d\Omega}{32\pi m_\pi^2}$$

we note that $p \rightarrow \frac{m_\pi}{2}$ since $E_p = \sqrt{p^2 + \cancel{m^2}} = p$

and $2E_p = E_\pi = m_\pi \Rightarrow p = \frac{m_\pi}{2}$

Next, we note that because of the decay symmetry, to avoid double-counting we integrate half of the solid angle sphere.

$$\Gamma = \frac{\mathcal{M}}{64\pi^2 m_\pi}(2\pi) = \frac{\mathcal{M}}{32\pi m_\pi}$$

Subbing in the expression for $\mathcal{M}$:

$$\Gamma = \frac{1}{32\pi m_\pi}\left[\frac{2\alpha^2 m_\pi^4}{4\pi^2 F_\pi^2}\left(\frac{N_c}{3}\right)^2\right]$$

$$= \frac{\alpha^2 m_\pi^3}{(4\pi)^3 F_\pi^2}\left(\frac{N_c}{3}\right)^2$$

We solve for $N_c$ by comparing with $\tau = 8.52 \times 10^{-17}$ s.

$$\Gamma = \frac{1}{(4\pi)^3}\left(\frac{1}{137}\right)^2 \frac{(135\text{ MeV})^3}{(92\text{ MeV})^2}\ \frac{N_c^2}{9}$$

$$= 8.6719 \times 10^{-1}\text{ eV } N_c^2$$

$$= 1.31749 \times 10^{15}\text{ s}^{-1} N_c^2$$

$$\Rightarrow \tau = 0.75902 \times 10^{-15}\text{ s } N_c^{-2} \simeq 8.52 \times 10^{-17}\text{ s}$$

$$\Rightarrow N_c^2 \simeq 8.909 \rightarrow N_c = 3$$ quark colours (Red, Green, & Blue)