David St-Amant Farmer
PHYS 4E03

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Assignment #1: Solutions

1.1 $$\Delta x \Delta p \geqslant \frac{\hbar}{2} \Rightarrow \Delta p \geqslant \frac{\hbar}{2(\Delta x)_{nucleus}} = \frac{6.58 \times 10^{-22}\ MeV \cdot s}{2 \cdot (10^{-15}\ m)}$$

$$\Rightarrow \underline{\underline{P_{min}}} = 0.329\ \frac{eV \cdot s}{m}$$

Then, $\boxed{E_{Kin}} = \sqrt{(P_{min})^2 c^2 + (mc^2)^2} - (mc^2)$, $(mc^2)_{electron} = 511\ KeV$

$$= \sqrt{\left(0.329\ \frac{eV \cdot s}{m}\right)^2 c^2 + (511 \times 10^3\ eV)^2} - 511 \times 10^3\ eV$$

$$\approx 9.81 \times 10^7\ eV = \boxed{9.81 \times 10^4\ KeV}$$

We see that $E_{Kin} \gg E_\beta \approx 17\ KeV$, so β decay cannot be caused by electrons being ejected from the nucleus.

$\boxed{1.2}$ We want $x'^{\mu} = \Lambda^{\mu}{}_{\alpha}(u_y)\,\Lambda^{\alpha}{}_{\nu}(v_x)\,x^{\nu}$, let $c=1$

$$\Lambda(v_x) = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \gamma = \frac{1}{\sqrt{1-v^2}}$$

$$\Lambda(u_y) = \begin{pmatrix} \tilde{\gamma} & 0 & -\tilde{\gamma}u & 0 \\ 0 & 1 & 0 & 0 \\ -\tilde{\gamma}u & 0 & \tilde{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \tilde{\gamma} = \frac{1}{\sqrt{1-u^2}}$$

$$\Rightarrow \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \tilde{\gamma} & 0 & -\tilde{\gamma}u & 0 \\ 0 & 1 & 0 & 0 \\ -\tilde{\gamma}u & 0 & \tilde{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$= \begin{pmatrix} \gamma\tilde{\gamma} & -\gamma\tilde{\gamma}v & -\tilde{\gamma}u & 0 \\ -\gamma v & \gamma & 0 & 0 \\ -\gamma\tilde{\gamma}u & \tilde{\gamma}\gamma uv & \tilde{\gamma} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

$$= \left.\begin{pmatrix} \gamma\tilde{\gamma}(t-vx) - \tilde{\gamma}uy \\ \gamma(x-vt) \\ \tilde{\gamma}\gamma u(vx-t) + \tilde{\gamma}y \\ z \end{pmatrix}\right\}$$

or let $u \to -u$, $v \to -v$ for a boost in $-x$ and $-y$ (Depending on your choice of direction)

$\boxed{1.3}$ We have $E = -u_\mu p^\mu = -\eta_{\mu\nu} u^\mu p^\nu$, $\eta_{\mu\nu} = \mathrm{diag}(-,+,+,+)$

This holds in all frames since E is a scalar, so pick the frame in which $u^\mu = \{1,0,0,0\}$, so we are in the rest frame of the particle, $p^\nu = \{m, \vec{p}\}$

$$\Rightarrow \underline{\underline{E}} = -(-1)(1)(m) + \vec{0}\cdot\vec{p} = m \overset{\text{restore } c}{\doteq} \underline{\underline{mc^2}}$$

So E is the energy of the particle.

In an arbitrary frame, we can boost to a different frame via $u'^\mu = \Lambda^\mu{}_\alpha u^\alpha$, $p'^\nu = \Lambda^\nu{}_\beta p^\beta$

$$\Rightarrow \underline{\underline{E}} = -\eta_{\mu\nu} u'^\mu p'^\nu = -\eta_{\mu\nu}(\Lambda^\mu{}_\alpha u^\alpha)(\Lambda^\nu{}_\beta p^\beta)$$

$$= -(\Lambda^\mu{}_\alpha \eta_{\mu\nu} \Lambda^\nu{}_\beta) u^\alpha p^\beta$$

$$= -(\Lambda^T \eta \Lambda)_{\alpha\beta} u^\alpha p^\beta$$

$$= -(\eta_{\alpha\beta}) u^\alpha p^\beta, \text{ by eq (1.50)}$$

$$= \underline{\underline{-\eta_{\mu\nu} u^\mu p^\nu}}, \text{ by relabeling } \alpha \to \mu, \beta \to \nu$$

$\Rightarrow$ E is the same in any frame, so it is Lorentz invariant.

$\boxed{1.4}$ a) We have $X^\mu(u) = \{l\sinh(\alpha u), 0, 0, l\cosh(\alpha u)\}$

① We want τ, so use $d\tau^2 = -ds^2 = -\eta_{\mu\nu}dx^\mu dx^\nu$

$$\Rightarrow dx^\mu = \left(\frac{\partial x^\mu}{\partial u}\right)du = \{\alpha l\cosh(\alpha u), 0, 0, \alpha l\sinh(\alpha u)\}du$$

$$\Rightarrow d\tau^2 = -(-1)(\alpha l\cosh(\alpha u))^2 du^2 - (\alpha l\sinh(\alpha u))^2 du^2$$

$$= (\alpha l)^2 \underbrace{(\cosh^2(\alpha u) - \sinh^2(\alpha u))}_{=1} du^2 = (\alpha l du)^2$$

$$\Rightarrow \underbrace{d\tau = \alpha l du}_{(*)}, \text{ so } \underline{\tau = \alpha l u} + \cancel{C} \quad \text{set to } 0$$

Thus, since $x^0 = t = l\sinh(\alpha u) \Rightarrow \underline{u = \frac{1}{\alpha}\operatorname{arcsinh}\left(\frac{t}{l}\right)}$

$$\Rightarrow \boxed{\tau = l\operatorname{arcsinh}\left(\frac{t}{l}\right)}$$

② We want $u^\mu = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{du}\left(\frac{du}{d\tau}\right) = \frac{dx^\mu}{du}\left(\frac{1}{\alpha l}\right)$, from $(*)$

$$= \boxed{\{\cosh(\alpha u), 0, 0, \sinh(\alpha u)\}}$$

③ Then, $\boxed{p^\mu = m u^\mu}$

④ Finally, $a^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{d}{d\tau}(u^\mu) = \frac{du^\mu}{du}\frac{du}{d\tau}$

$$= \frac{1}{\alpha l}\{\alpha \sinh(\alpha u), 0, 0, \alpha \cosh(\alpha u)\}$$

$$= \boxed{\frac{1}{l}\{\sinh(\alpha u), 0, 0, \cosh(\alpha u)\}}$$

l is initial position z at $u=0$ $(t=0)$

$$= \boxed{\frac{x^\mu}{l^2}}, \quad l \text{ scales } t \; (t = l\sinh(\alpha u))$$

α scales u $\left(u = \frac{1}{\alpha}\operatorname{arcsinh}\left(\frac{t}{l}\right)\right)$

↳ scales the acceleration

b) We have $x^\mu(t) = \{t, d\cos(wt), d\sin(wt), 0\}$, $\underline{\underline{c \equiv 1}}$

① $d\tau^2 = -\eta_{\mu\nu}\,dx^\mu dx^\nu$, $\quad dx^\mu = \left(\frac{\partial x^\mu}{\partial t}\right)dt$

$$= \{1, -wd\sin(wt), wd\cos(wt), 0\}\,dt$$

$$\Rightarrow d\tau^2 = \left[-(-1) - (wd)^2\left(\sin^2(wt) + \cos^2(wt)\right)\right]dt^2$$

$$= \left(1 - (wd)^2\right)dt^2 \equiv \gamma^{-2}dt^2$$

$$\Rightarrow \boxed{\tau = t/\gamma}, \quad \gamma \equiv \frac{1}{\sqrt{1-(wd)^2}}$$

② Then, $\boxed{u^\mu} = \frac{dx^\mu}{d\tau} = \frac{dx^\mu}{dt}\left(\frac{dt}{d\tau}\right) = \frac{dx^\mu}{dt}\gamma$

$$= \boxed{\gamma\{1, -wd\sin(wt), wd\cos(wt), 0\}}$$

③ We simply have $\boxed{p^\mu = mu^\mu}$

④ Finally, $a^\mu = \frac{d}{d\tau}\left(u^\mu\right) = \frac{du^\mu}{dt}\frac{dt}{d\tau}$

$$= \gamma^2\{0, -w^2 d\cos(wt), -w^2 d\sin(wt), 0\}$$

$$= \boxed{-\gamma^2 w^2 d\{0, \cos(wt), \sin(wt), 0\}}$$

w is the angular velocity and d is the orbital radius, so (wd) is the tangential velocity $wd < c$.

1.5 We have light rays with frequency w with $K^\mu = \{\hbar w_*, \hbar w_* \cos(\theta), \hbar w_* \sin(\theta), 0\}$ as their 4-momentum

① $$\underline{K_\mu K^\mu} = \eta_{\mu\nu} K^\mu K^\nu = -(\hbar w_*)^2 + (\hbar w_*)^2 \Big(\underbrace{\cos^2(\theta) + \sin^2(\theta)}_{1}\Big)$$
$$= \underline{0}$$

② From 1.4 a): $U^\mu = \{\cosh(\alpha u), 0, 0, \sinh(\alpha u)\}$, $E = -\eta_{\mu\nu} U^\mu p^\nu$

$$\Rightarrow E = -(-1)\cosh(\alpha u)(\hbar w_*) - \sinh(\alpha u)(0) = \underbrace{\hbar w_* \cosh(\alpha u)}_{\text{In frame of observer with } U^\mu}$$

$= \hbar w$ } In photons' rest frame

$\Rightarrow \boxed{w_* = w \operatorname{sech}(\alpha u)}$, where for $\alpha, u \geq 0$ $\;0 < \operatorname{sech}(\alpha u) < 1$

③ From 1.4 b): $U^\mu = \gamma\{1, -wd\sin(wt), wd\cos(wt), 0\}$, $\gamma = \frac{1}{\sqrt{1-(wd)^2}}$

$$\Rightarrow E = -(-1)(\gamma)(\hbar w_*) - (\hbar w_*)(\gamma w d)\underbrace{\left(-\sin(wt)\cos(\theta) + \cos(wt)\sin(\theta)\right)}_{= \sin(\theta - wt)}$$

$$= \hbar \gamma w_* [1 - \underbrace{wd}_{<1} \underbrace{\sin(\theta - wt)}_{|\cdot| \leq 1}] \Rightarrow \boxed{w_* = \frac{w}{\gamma(1 - wd\sin(\theta - wt))}}$$

and $E = \hbar w$