Physics 3C3

Term Test — Prof. D.W.L. Sprung

Duration of Examination: 2 Hours — 6:30 p.m — Thursday March 6, 2008

This examination paper includes 4 questions on 3 pages.

NOTE: Complete answers to **three** questions will constitute a complete paper, but all work submitted will be graded. (That means that unless you cross something out I will look at it, and base your mark on the best three questions.)

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1. For a system of N point particles in an inertial frame of reference, the center of mass position R and momentum P are defined by

$$M = \sum_i m_i \, ; \qquad M\vec{R} = \sum_i m_i \, \vec{r}_i \, ; \qquad M\vec{V} = \sum_i m_i \, \vec{v}_i \equiv \vec{P} \; .$$

Let $r_i - R = r_i'$ define coordinates "in the centre of mass (CM) system". Let $\vec{F}_i$ be the external force acting on r_i, and $\vec{F}_{ji}$ be the internal force of particle j acting on i.

(a) **Show** that the total angular momentum and the kinetic energy divide into discrete parts

$$\vec{L} = \vec{L}_{CM} + \vec{L}' \; ; \qquad T = T_{CM} + T' \; .$$

where

$$\vec{L}_{CM} = \vec{R} \times \vec{P} \; ; \qquad T_{CM} = \frac{1}{2} M \dot{R}^2$$

are associated with the motion of the center of mass of the system, and the primed quantities describe internal motion around the center of mass.

(b) Show that the rate of change of the internal angular momentum $\vec{L}'$ depends only on the torques due to the external forces, acting around the center of mass:

$$\frac{d\vec{L}'}{dt} = \sum_i \vec{r}_i' \times \vec{F}_i \; .$$

(This assumes that the internal forces act along the line joining the particles.)

(c) Explain Newton's concept of an inertial coordinate system in classical mechanics. Give some examples of coordinate systems that can be considered inertial and some that are not.

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2. Consider motion of a particle in a rotating coordinate system with basis vectors $\hat{e}_i$ (body-fixed axes), as compared to an inertial frame $\hat{e}_i^{(0)}$ having the same origin. An arbitrary vector $\vec{V}$ can be described in two ways:

$$\vec{V} = \sum_i V_i^{(0)} \, \hat{e}_i^{(0)} = \sum_i V_i \, \hat{e}_i$$

(a) Justify the relations

$$\Big(\frac{d\vec{V}}{dt}\Big)_{in} \;=\; \Big(\frac{d\vec{V}}{dt}\Big)_{body} + \sum_i V_i \frac{d\hat{e}_i}{dt} = \Big(\frac{d\vec{V}}{dt}\Big)_{body} + \vec{\omega} \times \vec{V} \;, \tag{1}$$

where $\vec{\omega}$ is the instantaneous angular velocity vector, which points along the instantaneous axis of rotation.

(b) For motion in a rotating frame of reference, justify the relation

$$\Big(\frac{d^2\vec{r}}{dt^2}\Big)_{in} \;=\; \Big(\frac{d^2\vec{r}}{dt^2}\Big)_{body} + 2\big(\vec{\omega} \times \vec{v}\big) + \Big(\frac{d\vec{\omega}}{dt} \times \vec{r}\Big) + \big(\vec{\omega} \times (\vec{\omega} \times \vec{r})\big) = F_{ext}/m \;, \tag{2}$$

where F_{ext} is the applied force and m is the mass involved.

(c) Identify the Coriolis and centrifugal accelerations in eq. 2, and describe some examples of the effects they have for motion as seen by terrestrial observers.

3. Hamilton's principle states that the classical action

$$S = \int_A^B L(q, \dot{q}; t) dt$$

between fixed end-points, is stationary under small variations in the path

$$[q_s(t) \longrightarrow q_s(t) + \delta q_s(t)] \;.$$

(a) Explain what that means. Is S ever a maximum? Does it matter whether it is a minimum or a stationary point?

(b) Consider making such a variation in the path. Show that the velocity changes as follows:

$$\dot{q}_s(t) \longrightarrow \dot{q}_s(t) + \frac{d\delta q_s(t)}{dt} \;.$$

(c) Use Hamilton's principle and calculus of variations to deduce Lagrange's equations

$$\frac{d}{dt}\Big[\frac{\partial L(q,\dot{q})}{\partial \dot{q}_s}\Big] - \frac{\partial L(q,\dot{q})}{\partial q_s} = 0 \;.$$

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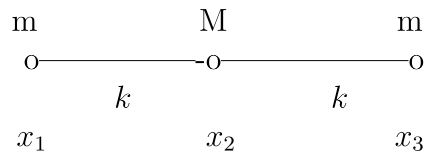
4. Consider linear oscillations of a triatomic molecule, containing a central atom of mass M and two satellites of mass m, equidistant on either side. Take the spring constant for their interaction to be k.



(a) Write down the Lagrangian for this system, taking the displacements from equilibrium to be x_1, x_2, x_3. Show that the equations for normal modes are

$$\begin{pmatrix} k - m\omega^2 & -k & 0 \\ -k & 2k - M\omega^2 & -k \\ 0 & -k & k - m\omega^2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \ .$$

(b) Show that the eigenfrequencies are

$$\omega_1^2 = 0 \ , \qquad \omega_2^2 = \frac{k}{m} \ , \qquad \omega_3^2 = \frac{k}{m} + \frac{2k}{M} \ ,$$

with normal mode vectors (columns of the modal matrix)

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} , \qquad \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} , \qquad \begin{pmatrix} 1 \\ -2m/M \\ 1 \end{pmatrix}$$

respectively. (You may simply verify the result, or do it constructively.)

(c) Describe in words the oscillation patterns of the atoms corresponding to the second and third normal modes. Is it reasonable that the highest frequency corresponds to the third mode?

(d) What is the physical significance of the zero-frequency eigen-mode? Should we have foreseen it? Is it really a normal mode?

Miscellaneous information

$$\begin{aligned} (\vec{A} \times \vec{B})_3 &= A_1 B_2 - A_2 B_1 \qquad \text{and cyclically} \\ \cos(A+B) &= \cos A \ \cos B - \sin A \ \sin B \\ \sin(A+B) &= \sin A \ \cos B + \cos A \ \sin B \\ \cos 2x &= 1 - 2\sin^2 x = 2\cos^2 x - 1 \end{aligned} \qquad (3)$$

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