Physics 3C3

Term Test — Prof. D.W.L. Sprung

Duration of Examination: 2 Hours — 3:30 p.m — Friday March 5, 2010

This examination paper includes 4 questions on 3 pages.

NOTE: Complete answers to **three** questions will constitute a complete paper, but all work submitted will be graded. (That means that unless you cross something out I will look at it, and base your mark on the **best three** questions.)

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1. Hamilton's principle states that the classical action between fixed end-points,

$$S = \int_A^B L(q, \dot{q}; t) dt \ ; \qquad L = T - V \ ,$$

is stationary under small variations in the path $q_s(t) \to q_s(t) + \delta q_s(t)$. (a) Explain what that means. Is S ever a maximum? Does it really matter whether it is a maximum, a minimum or a stationary point?

(b) Consider making such a variation in the path. Changing $q_s(t)$ forces a change in the associated velocity. Show that the velocity changes as follows:

$$\dot{q}_s(t) \to \dot{q}_s(t) + \frac{d\delta q_s(t)}{dt} \ .$$

(c) Derive Lagrange's equations of motion from Hamilton's principle

$$\frac{d}{dt}\Big[\frac{\partial L(q,\dot{q})}{\partial \dot{q}_s}\Big] - \frac{\partial L(q,\dot{q})}{\partial q_s} = 0 \ .$$

(If you find it too complicated to have $s > 1$ degrees of freedom, try first with $s = 1$.)

Sketch of Foucault pendulum
(copied from Fig. 12.1, page 45 of text)

Continued overleaf ...

2. Consider motion in a rotating coordinate system with basis vectors $\hat{e}_i$ (body-fixed axes), as compared to an inertial frame $\hat{e}_i^{(0)}$ having the same origin. An arbitrary vector $\vec{V}$ can be described in two ways:

$$\vec{V} = \sum_i V_i^{(0)} \, \hat{e}_i^{(0)} = \sum_i V_i \; \hat{e}_i$$

(a) Justify the relation

$$\begin{aligned} \Big(\frac{d\vec{V}}{dt}\Big)_{in} &= \Big(\frac{d\vec{V}}{dt}\Big)_{body} + \sum_i V_i \frac{d\hat{e}_i}{dt} \\ &= \Big(\frac{d\vec{V}}{dt}\Big)_{body} + \vec{\omega} \times \vec{V} \; , \end{aligned} \qquad (1)$$

where $\vec{\omega}$ is the instantaneous angular velocity vector, pointing along the axis of rotation.

(b) For motion on the surface of the earth justify the relation

$$\Big(\frac{d^2r}{dt^2}\Big)_{in} = \Big(\frac{d^2r}{dt^2}\Big)_{body} + 2\Big(\vec{\omega} \times \vec{v}\Big) + \Big(\dot{\omega} \times \vec{r}\Big) + \Big(\vec{\omega} \times (\vec{\omega} \times \vec{r})\Big) = \vec{F}_{ext}/m \; , \qquad (2)$$

where F_{ext} are the applied forces, and all positions r and velocities v are vectors.

(c) The Foucault pendulum provides a striking demonstration that the earth rotates on its axis. Take a coordinate system as indicated in the sketch on page 1, with the $\hat{z}$-axis vertical, the $\hat{x}$-axis pointing south and the $\hat{y}$-axis pointing east, at our location. The angular velocity of the earth is a vector

$$\vec{\omega} = \omega(-\sin\theta, \, 0, \, \cos\theta)$$

where θ is our co-latitude. Let $\vec{T}$ be the tension in the cord, and ℓ its length. Write out the equation of motion of the mass m:

$$m\ddot{r} = m\vec{g} + \vec{T} - 2m(\vec{\omega} \times \dot{r})$$

in Cartesian components. Justify the approximation $T\cos\psi \sim mg$.

(d) Show that the equations of motion for the horizontal components can be combined in the form $\zeta(t) = x(t) + iy(t)$, to give

$$\ddot{\zeta} + 2i\omega \, \cos\theta \, \dot{\zeta} + \frac{g}{\ell}\zeta = 0 \; .$$

From there show that the plane of oscillation rotates clockwise at angular rate $\omega\cos\theta$, making the period about 36 hours.

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3. In the scattering of a point particle from a spherical fixed target, the orbit is determined by the initial velocity v_∞ and initial position $(-\infty, b)$, where b is the "impact parameter". (a) How are the parameters b and v_∞ related to the energy and angular momentum of the particle? Under what conditions are E and $\vec{L}$ conserved? What property of the potential ensures that motion proceeds in a plane?

(b) An instructive example of scattering, where the orbit requires no calculation, is scattering from a hard sphere of radius R. (No force acts until the projectile reaches the sphere; then it will reflect from the sphere according to the rule "angle of incidence = angle of reflection".) Show that the scattering angle is given by

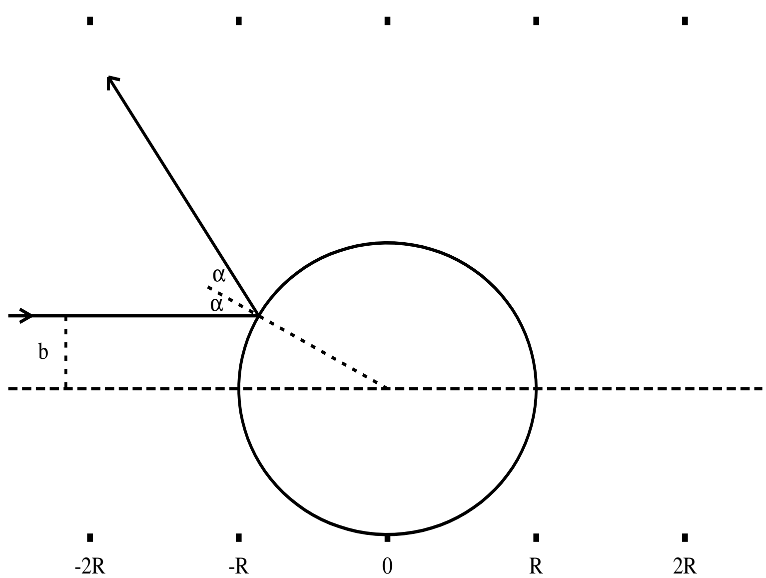
$$\cos(\theta/2) = b/R \ . \tag{3}$$

What range of scattering angles is possible? Verify that the angular momentum of the scattered particle is unchanged.

(c) **Define** the differential scattering cross-section and show that for this problem it is

$$\sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \frac{R^2}{4} \ . \tag{4}$$

How do you interpret the total cross-section in this example?



(d) By extending the final trajectory backwards, verify that the final angular momentum is the same as the initial.

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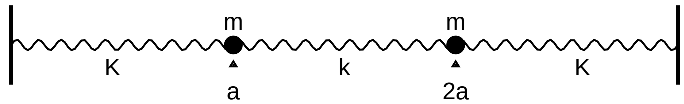
4. Consider linear oscillations of a pair of equal point masses m connected by a spring of force constant k, held between two fixed walls located at $x = 0, 3a$ by springs (constant K).



(a) Write down the Lagrangian for this system, taking the displacements from equilibrium to be $q_j = x_j - ja$, $j = 1, 2$. Show that the equations for the normal modes are

$$\begin{pmatrix} K+k-m\omega^2 & -k \\ -k & K+k-m\omega^2 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = 0 \ .$$

(b) Show that the eigenfrequencies are given by

$$m\omega_1^2 = K \ , \qquad m\omega_2^2 = K + 2k \ ,$$

with normal mode vectors (columns of the modal matrix $\mathcal{A}$)

$$\frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \ , \qquad \frac{1}{\sqrt{2m}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ .$$

respectively. (You may simply verify the result, or do it constructively.)

(c) Describe in words the oscillation patterns of the masses corresponding to each of the normal modes. Why does the lowest frequency correspond to the symmetric mode?

(d) What happens if the outer springs are removed ($K \to 0$)? Is a zero-frequency mode an oscillatory motion, or if not, what does it mean? How is this situation related to Newton's first law?

Miscellaneous information

$$\begin{aligned} (\vec{A} \times \vec{B})_3 &= A_1 B_2 - A_2 B_1 \qquad \text{and cyclically} \\ \cos(A+B) &= \cos A \ \cos B - \sin A \ \sin B \\ \sin(A+B) &= \sin A \ \cos B + \cos A \ \sin B \\ \cos 2x &= 1 - 2\sin^2 x = 2\cos^2 x - 1 \end{aligned} \tag{5}$$

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