Physics 3C03, 6C03

Day Class | Prof. D.W.L. Sprung

Duration of Examination: 3 Hours

McMaster University Final Examination | April 24, 2008

Student Name: **Student Number**:

Please read the paper before you decide where to begin. This examination paper includes 5 questions on 5 pages. You are responsible for ensuring that your copy of the paper is complete. Please bring any discrepancy to the attention of an invigilator.

NOTE: Candidates may bring and use the Special Examination Memorandum Sheet. The *standard* McMaster Pocket calculator (Casio *fx* 991) is also permitted. Your mark will be based on the best **four** questions completed, but all work submitted will be graded.

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1. Take a coordinate system with OZ pointing vertically downwards. A circular wire ring of radius a hangs in a vertical plane, with its centre at the origin. The plane rotates around OZ at constant angular velocity $\dot{\phi} = \Omega$. (The dot means the time derivative d/dt.) A bead of mass m moves without friction on the wire.

(a) Construct the Lagrangian for the motion of the bead, taking the polar angle θ as the generalized coordinate.

$$L = \frac{1}{2} m a^2 \left[\dot{\theta}^2 + \Omega^2 \sin^2\theta \right] + mga\cos\theta \ .$$

(b) Show that Lagrange's equation is

$$\ddot{\theta} + \left[\frac{g}{a} - \Omega^2 \cos\theta \right] \sin\theta = 0 \ .$$

Also show that the bead is in equilibrium at angle θ_0 where $\cos\theta_0 = g/(a\,\Omega^2)$. Verify this result by balancing the centrifugal acceleration against gravity in a rotating coordinate system.

(c) Show that the bead executes simple harmonic motion, for small displacements from the equilibrium angle, $\theta = \theta_0 + \eta$, according to

$$\ddot{\eta} + (\Omega^2 \sin^2\theta_0)\ \eta = 0 \ .$$

(d) What happens for very slow rotational speeds $\Omega^2 < \Omega_0^2 \equiv g/a$? Show in this case that small oscillations are governed by

$$\ddot{\eta} + (\Omega_0^2 - \Omega^2)\ \eta = 0 \ .$$

Describe in words the behaviour of the bead in the three regimes of Ω^2.

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2. A different problem, relating to a bead of mass m moving freely on a horizontal wire hoop of radius a, leads to the following Lagrangian, which is purely kinetic energy:

$$T = L(\phi, \dot{\phi}) = \frac{ma^2}{2} \left[\dot{\phi}^2 + 2\omega\, \dot{\phi}\, \cos(\phi - \omega t) \right] \ .$$

The angle ϕ gives the position of the bead, while ω is a constant angular velocity of the hoop.

(a) Since the Lagrangian depends explicitly on time, the momentum will not be simply $m\dot{\phi}$. **Show** that the canonical momentum $p = p_\phi \equiv \partial L / \partial \dot{\phi}$ conjugate to ϕ is given by

$$\frac{p}{ma^2} = \dot{\phi} + \omega \cos(\phi - \omega t)) \ .$$

(b) Complete the Lagrange equation to obtain the equation of motion

$$\ddot{\phi} + \omega^2 \sin(\phi - \omega t) = 0 \ .$$

(Explanatory note: If we define $\theta = \phi - \omega t$, this would be the equation of the physical pendulum: θ would be the bead angle measured in a rotating coordinate system. The whole problem could be worked out in terms of θ but it is much messier.)

(c) Using (a), deduce that the Hamiltonian is

$$H(p, \phi) = p\dot{\phi} - L = \frac{ma^2}{2}\, \dot{\phi}^2 = \frac{ma^2}{2} \left[\frac{p}{ma^2} - \omega \cos(\phi - \omega t) \right]^2 \ .$$

(d) Verify that Hamilton's equations

$$\dot{\phi} = \frac{\partial H}{\partial p} \ ; \qquad \dot{p} = -\frac{\partial H}{\partial \phi} \ ,$$

lead to the same equation of motion as in part (b).

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3. In discussing the motion of a rigid body, it is convenient to introduce an inertial frame, and a body-fixed frame of reference, having the same origin. Euler's equations follow from the statement

$$\left(\frac{d\vec{L}}{dt}\right)_{in} \;=\; \left(\frac{d\vec{L}}{dt}\right)_{body} + \vec{\omega} \times \vec{L} \;=\; \vec{\Gamma}\,, \tag{1}$$

where $\vec{\omega}$ is the instantaneous angular velocity vector, and $\vec{L}$ is the angular momentum.

(a) What does $\vec{\Gamma}$ represent? From eq. 1, derive Euler's equations, which describe the motion in the body-fixed frame.

(b) Take the rigid body to be collection of mass points m_p at positions $\vec{r}_p$. Show that its kinetic energy can be expressed as a matrix product $2T = \vec{\omega}^T \, \hat{I} \, \vec{\omega}$ where $\hat{I}$ is the inertia tensor whose elements are

$$I_{ij} = \sum_p m_p \left(r_p^2 \delta_{i,j} - x_{pi} x_{pj}\right) \,,$$

and x_{pi} is the i'th Cartesian component of the vector $\vec{r}_p$, and δ_{ij} is the Kronecker symbol.

(c) Similarly, show that the angular momentum vector can be expressed component-wise as

$$L_i = \sum_j \hat{I}_{ij} \omega_j \qquad \text{or} \quad \vec{L} = \hat{I}\vec{\omega}\;.$$

Explain what the principal axes of inertia are, and why they are useful. Give an example of a rigid body where it is easy to identify the principal axes.

(d) The earth is, to a good approximation, an oblate spheroid of rotation, with $I_3 > I_1 = I_2$. From Euler's equations, with no external torques, show that ω_3 is a constant of motion. Show further that the other two components of $\vec{\omega}$ obey coupled equations of motion which can be put in the form

$$\dot{\omega_1} \;=\; -\Omega\,\omega_2\,, \qquad \dot{\omega_2} = \Omega\,\omega_1\,, \quad \text{where} \qquad \Omega \equiv \omega_3 \frac{I_3 - I_1}{I_1} \sim \frac{\omega_3}{305}\;. \tag{2}$$

Solve these equations for $\vec{\omega}(t)$, and describe in words the motion of the angular velocity vector with respect to the symmetry axis of the earth, as seen by an earth-bound observer.

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4. Hamilton's Principle states that the action integral is stationary against small variations δq_i of the path followed by a mechanical system between fixed end points at times t_1 and t_2. When this is applied to the Hamiltonian we have to consider arbitrary variations $\delta q_i(t)$ and $\delta p_i(t)$ of the coordinates and momenta, and require that

$$\delta \int_1^2 L(q, \dot{q}, t) dt = \delta \int_1^2 [\sum_i p_i \, \dot{q}_i - H(q, p, t)] dt = 0 \; .$$

(q, p without indices refers to the entire set of coordinates and momenta.)
(a) Show that this leads to Hamilton's equations of motion for each pair of conjugate variables:

$$\dot{q}_j = \frac{\partial H}{\partial p_j} \; ; \qquad \dot{p}_j = -\frac{\partial H}{\partial q_j} \; .$$

Suggestion: You need to use the identity $p_j \, \delta \dot{q}_j = \frac{d}{dt}[p_j \, \delta q_j] - \dot{p}_j \, \delta q_j$.
(b) Consider making a canonical change of variables from the set $\{q, p\}$ to a new set $\{Q, P\}$ with $Q_i = Q_i(q, p)$, and $P_i = P_i(q, p)$. Show that if

$$\sum_i p_i \dot{q}_i - H(q, p, t) = \sum_j P_j \dot{Q}_j - \tilde{H}(Q, P, t) + \frac{dF(q, Q, t)}{dt}$$

then Hamilton's principle implies that $\tilde{H}$ plays the role of the new Hamiltonian and Hamilton's equations of motion apply in the new variables. ($F(q, Q, t)$ is called a type-one generator of the canonical transformation.)
As part of the demonstration, you will want to impose the following relations:

$$p_s = \frac{\partial F}{\partial q_s}(q, Q, t) \; ; \qquad -P_s = \frac{\partial F}{\partial Q_s}(q, Q, t) \; ; \qquad H(q, p, t) = \tilde{H}(Q, P, t) - \frac{\partial F}{\partial t} \; .$$

(c) Explain how Hamilton-Jacobi theory leads to the solution of a mechanical problem by using a type-two generating function $S(q, P, t)$ (which is called Hamilton's principal function):

$$F \equiv S(q, P, t) - \sum_j P_j Q_j \; .$$

Suggestion: In this case you will want to impose

$$p_i = \frac{\partial S}{\partial q_i} \; ; \qquad Q_i = \frac{\partial S}{\partial P_i} \; ; \qquad \tilde{H}(Q, P, t) = H(q, p, t) + \frac{\partial S}{\partial t} \; .$$

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5. Consider a system of independent harmonic oscillators for which

$$H(q,p) = \sum_i \left[\frac{p_i^2}{2m_i} + \frac{1}{2}k_i q_i^2\right] = \sum_i H_i(q_i,\, p_i)\ .$$

Since the Hamiltonian has no explicit time-dependence, it represents the energy. The system is separable, so we can apply the Hamilton-Jacobi method in the form

$$S(q,P,t) = W(q,P) - Et = \sum_s W_s(q_s,\, p_s) - \alpha_s t\ .$$

(a) From the Hamilton-Jacobi equation, show that

$$p_s = \frac{\partial W}{\partial q_s} = \pm\sqrt{2m_s[\alpha_s - (k_s/2)q_s^2]}\ .$$

(b) Show that the action $J_s \equiv \oint p_s\, dq_s = 2\pi\alpha_s\sqrt{m_s/k_s} \equiv \alpha_s/\nu_s$.
(c) Express the Hamiltonian in the action-angle representation.
(d) In the Bohr-Sommerfeld method of quantisation, it is postulated that the action takes only the values $J_s = n_s h$, where h is Planck's constant and $n_s = 0,\, 1,\, \cdots$ is a non-negative integer. Write down a list of the low-lying energy levels of the quantised three-dimensional oscillator with up to three quanta of energy.

The END

Canonical transformations

Generating function of type 1: $F = F_1(q,Q,t)$.
Derivatives:

$$p_i = \frac{\partial F_1}{\partial q_i} \qquad P_i = -\frac{\partial F_1}{\partial Q_i}$$

Trivial special case: $F_1 = q_i\, Q_i,\ Q_i = p_i,\ P_i = -q_i$.

Generating function of type 2: $F = F_2(q,P,t) - Q_i\, P_i$.
Derivatives:

$$p_i = \frac{\partial F_2}{\partial q_i} \qquad Q_i = \frac{\partial F_2}{\partial P_i}$$

Trivial special case: $F_2 = q_i\, P_i,\ Q_i = q_i,\ P_i = p_i$.

Generating function of type 3: $F = F_3(p,Q,t) + q_i\, p_i$.
Derivatives:

$$q_i = -\frac{\partial F_3}{\partial p_i} \qquad P_i = -\frac{\partial F_3}{\partial Q_i}$$

Trivial special case: $F_3 = p_i\, Q_i,\ Q_i = -q_i,\ P_i = -p_i$.

Generating function of type 4: $F = F_4(p,P,t) + q_i\, p_i - Q_i\, P_i$.
Derivatives:

$$q_i = -\frac{\partial F_4}{\partial p_i} \qquad Q_i = \frac{\partial F_4}{\partial P_i}$$

Trivial special case: $F_4 = p_i\, P_i,\ Q_i = p_i,\ P_i = -q_i$.