2007 Canadian Association of Physicists Prize Examination

Tuesday, February 6, 2007

Duration: 3 hours

Instructions:

You are permitted to use calculators for the exam.

There are a total of 10 questions on 11 pages.

Each question will be marked by a different examiner: **the answer to each question should be written on a separate page.** If more than one page is required for any question, then those pages should be stapled together separate from other questions.

The number of the question, the name of the candidate, and the name of the university and department should be clearly indicated on the first page of each answer.

Each question has equal value. You are not expected to attempt all the questions! Relax and attempt the questions on material that you are most familiar with or those questions that just look the most interesting to you.

The completed examination papers should be sent by the Department Chairpersons to:

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The examining committee members are:

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QUESTION 1

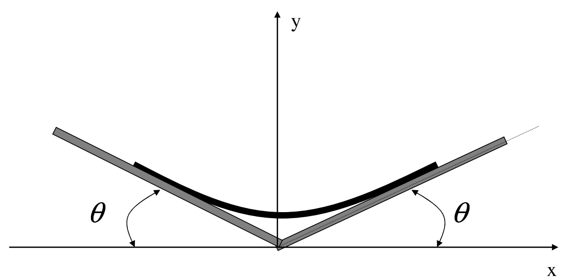
A rope of length L and mass m rests on two platforms which are both inclined at an angle θ (which you are free to pick), as shown below. The rope has a uniform mass density, and its coefficient of friction with the platforms is μ_s=1. The system is symmetrical about the y-axis.

(a) Draw a free-body diagram for the rope. (Hint: Consider the fraction of the rope that is not in contact with the platforms and the segments in contact with the platforms separately.)

(b) Given that the rope is in equilibrium, write down Newton's 1^{st} law in component form.

(c) Using your results from part (b), find an expression in terms of θ for the fraction of the rope f that does not touch the platforms.

(d) For what angle θ is the fraction f of rope not touching the surface maximized?



QUESTION 2

The free-particle Schrödinger equation in three dimensions is given by:

$$-\frac{\hbar^2}{2m}\nabla^2\psi_k(\mathbf{r}) = \varepsilon_k\psi_k(\mathbf{r}). \tag{1}$$

(a) Assuming the solutions to equation (1) are plane waves:

$$\psi_k(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} \tag{2}$$

show that the energy is of the form:

$$\varepsilon_k = \frac{\hbar^2}{2m}(k_x^2 + k_y^2 + k_z^2). \tag{3}$$

(b) If $\psi_k(\mathbf{r})$ is periodic in x, y, and z with period L, what is the condition on the vector $\mathbf{k}$?

(c) In quantum mechanics, the momentum operator is given by:

$$\mathbf{p} = -i\hbar\vec{\nabla}.$$

Using this find the velocity $\mathbf{v}$ of a particle in the state $\psi_k(\mathbf{r})$.

(d) In the ground state of a system of volume $V = L^3$ containing N free electrons, the occupied orbitals can be seen as points inside a sphere of radius k_F in $\mathbf{k}$ space. The surface of this sphere is known as the Fermi surface. Show that the radius of the Fermi sphere is given by:

$$k_F = \left(\frac{3\pi^2 N}{V}\right)^{\frac{1}{3}}.$$

(e) Use the result of the previous question to derive the Fermi energy ε_F (the energy at the Fermi sphere).

(f) Show that the density of states at the Fermi energy $D(\varepsilon_F)$ is given by:

$$D(\varepsilon_F) \equiv \left.\frac{dN}{d\varepsilon}\right|_{\varepsilon_F} = \frac{3N}{2\varepsilon_F}.$$

QUESTION 3

Consider a charged particle in a magnetic field $\mathbf{B} = (0, 0, B)$. Because of gauge symmetry, there are an infinite number of choices for the vector potential $\mathbf{A}$, such that $\nabla \times \mathbf{A} = \mathbf{B}$. This problem concerns two popular choices, the "Landau gauge",

$$\mathbf{A} = (-BY, 0, 0)$$

and the "circular gauge"

$$\mathbf{A} = (-BY/2, BX/2, 0).$$

The Hamiltonian for a free particle with electric charge q in a magnetic field is

$$H_0 = \frac{(\mathbf{P} - q\mathbf{A})^2}{2m}.$$

(a) In the Landau gauge, find two operators out the following which commute with H_0 (two operators A and B commute when $[A, B] = AB - BA = 0$):

$$P_x, P_y, P_z, J_z$$

where

$$[R_i, R_j] = [P_i, P_j] = 0$$

$$[R_j, P_k] = i\hbar\delta_{jk}$$

$$\mathbf{J} = \mathbf{R} \times \mathbf{P}$$

$$\mathbf{R} = (X, Y, Z).$$

(b) Do the operators which you found in part (a) commute with each other?

(c) Now repeat parts (a) and (b), except this time use the circular gauge.

(d) Multiple choice: Pick one of the following four choices and justify your answer with a short explanation.

The fact that their are two different, non-commuting sets of commuting operators that both commute with H_0 implies that

i. The eigenvalues of H_0 are positive.

ii. The eigenstates of H_0 are degenerate.

iii. H_0 has no bound states.

iv. The position of a particle cannot be determined.

QUESTION 4

The gravitational field of a black hole is so strong that nothing escapes it, not even light. Therefore, throwing something into a black hole is an irreversible process. As such, adding mass to a black hole increases its entropy.

(a) Using dimensional analysis, find a combination of the gravitational constant G, the mass of the black hole M, and the speed of light c that gives dimensions of length. This defines the characteristic length L of the black hole.

(b) It can be argued that the entropy of a black hole is $S \sim N_{\max} k_{\mathrm{B}}$, where k_{B} is Boltzmann's constant and $N_{\max}$ is the maximum number of particles that could have been used to create the black hole. To find $N_{\max}$, assume that the particles used to make the black hole are photons of wavelength $\sim L$, and that the energy of the black hole is $U = Mc^2$. Find an expression for S in terms of G, M, c, k_{B} and h (Planck's constant). S should increase with M.

(c) Assuming that volume is constant, find an expression for the heat capacity of the black hole in terms of G, M, c, k_{B} and h. You should find that the heat capacity is negative.

QUESTION 5

(a) A box of volume V contains N noninteracting identical particles at low density. The partition function for this system can be approximated as

$$Z(N,V) = \frac{(V/\lambda_{\text{th}}^3)^N}{N!}$$

where λ_{th} is the thermal de Broglie wavelength for such particles at that temperature. What would the partition function be if the particles were distinguishable?

(b) A volumes, V_1 is located a distance H above a second volume V_2. The volumes are connected by a narrow tube. Both are at temperature T and both are thin in the vertical direction. Together, the volumes contain N_{total} identical particles, each of mass m, at low density.

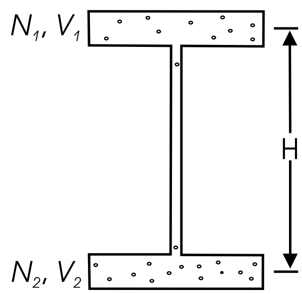


Figure 1.

i. What is the partition function, $Z(N_1, N_2)$, for the system when there are N_1 particles in V_1 and N_2 in V_2.

ii. At thermal equilibrium, what is the ratio of the particle densities in the two volumes? (Recall that the Helmholtz free energy for a system is $F = -kT \ln Z$).

iii. Consider the nonequilibrium situation for which the densities of particles in the upper and lower volumes are equal. What is the difference between the chemical potentials for particles in the two volumes? **Briefly** explain your answer. In which volume do particles have the higher chemical potential?

QUESTION 6

The density in an ocean of depth H varies with depth as $\rho = \rho_0(1 + cz)$, where ρ_0 is the density at the surface, c is a constant, and the z-axis is directed downwards. (Typically, the density varies due to different amounts of salt dissolved in the water at different depths. The lighter fluid is on top of heavier fluid providing for static stability of the fluid column).

(a) Find the distribution of hydrostatic pressure with depth.

(b) Find the total mass of a fluid column of unit horizontal cross sectional area and the position of its centre of mass. Is the center of mass above or below the middepth ($H/2$) of the column?

(c) If the water column is mixed (by waves, turbulence etc.), what is the density of the now uniform water? What is the change in potential energy of the water column?

(d) Suggest a density distribution that minimizes the potential energy of the column given that the total mass must be conserved and the density must vary between its minimum value ρ_0 at the surface and a maximum value at the bottom.

QUESTION 7

A bubble chamber contains a liquid that is kept very close to its boiling point. A large homogeneous magnetic field causes charged particles to curve one way or another. High energy charged elementary particles can give rise to micrometre-scale gaseous bubbles in the liquid along their trajectory - such a happening is termed an "event". Uncharged particles are invisible. The photo of the bubble chamber event shown is the decay of the kaon:

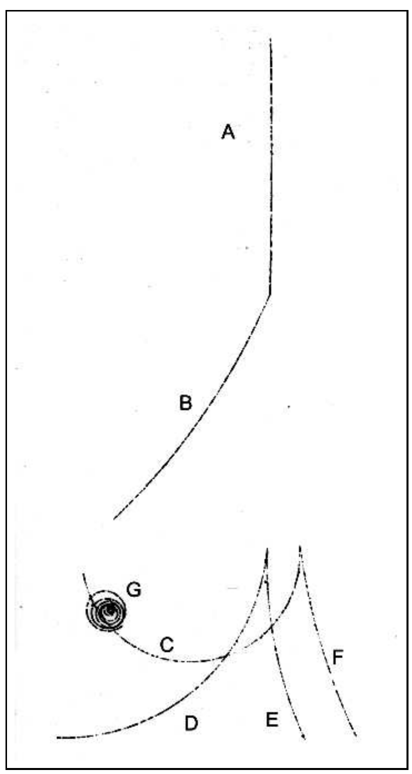
$$K^- \to \pi^- + \pi^0 \tag{1}$$

$$\pi^0 \to \gamma + \gamma \tag{2}$$

The gamma ray photons strike a lead sheet and then give rise to electron-positron pairs.

$$\gamma \to e^+ + e^- \tag{3}$$

(a) Identify the elementary particles that correspond to the following labeled tracks (hint - the loop labeled G is an extraneous electron which has been knocked out of an atom in the bubble chamber liquid and is not part of the event):
(b) What can you say from the figure, approximately, about the positron charge-mass ratio (sign and magnitude, as compared to that of the electron)? Provide a one-sentence justification for your answer.
(c) Which direction does the magnetic field point?
(d) An electron has rest mass $0.511 Mev/c^2$. What is the minimum energy for the gamma rays in the event?
(e) Gamma ray photons do not spontaneously give rise to electron-positron pairs in free space (hence the need for the lead sheet). Invoke special relativity to argue (in at most two sentences) why this is so.

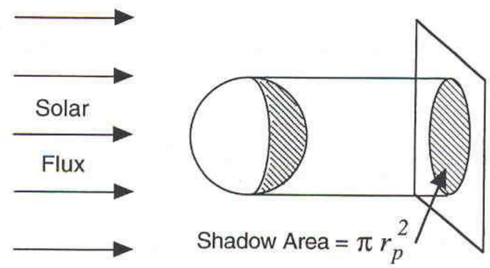


QUESTION 8

A black body with a temperature T emits an amount of radiation with flux density Υ (in W/m^2) according to

$$\Upsilon = \sigma T^4$$

where $\sigma = 5.67 \times 10^{-8}\, Wm^{-2}K^{-4}$. The solar constant is the solar radiation flux density at a distance r_e from the Sun ($r_e = 1$ A.U. = the mean radius of earth's orbit around the sun). For the upper boundary of the earth's atmosphere the solar constant is $S_0 = 1367\,\mathrm{W/m}^2$. Consider that the Earth absorbs only the proportion $(1-\alpha_p)$ of the solar radiation incident at the upper boundary of the atmosphere and assume that the Earth is a black body with a mean emission temperature T_e. Because the temperature of the Earth is neither increasing nor decreasing, the total terrestrial flux radiated to space must balance the solar radiation absorbed by the Earth.



(a) Show that:

$$\frac{S_0}{4}(1-\alpha_p) = \sigma T_e^4$$

(b) Calculate T_e for the typical values of the earth albedo $\alpha_p = 0.3$. The real mean temperature of the earth surface is $T_a = 289K$. Why are the values of T_a and T_e different?

(c) Jupiter has an internal heat source resulting from its gravitational collapse. Therefore the outgoing flux of planetary radiation per unit surface area, which is equal to σT_j^4 is balanced by this source and the solar radiation. The planetary radius of Jupiter is $R_j = 69500\,km$; its mean radius of orbit around the Sun is 5.19 A.U. (where 1 A.U. is the mean radius of the Earth's orbit); its planetary albedo is $\alpha_p = 0.51$, the measured emission temperature of Jupiter is $T_j = 130K$. Calculate the magnitude of Jupiter's internal heat source per unit volume.

QUESTION 9

The condition for sustainable modes in an optical fiber is that a wave which undergoes N internal reflections must interfere constructively with a wave that undergoes N±1 reflections, and this can only occur for certain common directions of propagation.

(a) Using the example of the parallel-sided planar waveguide shown below, justify the required interference condition:

$$\frac{\pi n_1 \gamma}{\lambda} + \delta_r = m\pi,$$

where n_1 is the refractive index of the core material, γ is the physical path difference, λ is the wavelength, δ_r is the phase change due to reflection, and m is an integer (*i.e.*, the mode number).

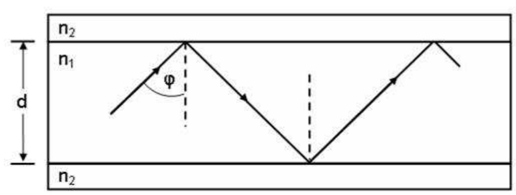
(b) Using the above relationship, show that each successful mode of propagation has an integer mode number m, related to a direction φ_m, given by

$$m \simeq \frac{2n_1 d \cos\varphi_m}{\lambda}.$$

(c) Finally, on the basis of the foregoing, derive the following expression for the maximum number of allowed modes:

$$m_{max} = \frac{2d}{\lambda}\sqrt{n_1^2 - n_2^2} + 1.$$

(d) If the refractive indices of the above waveguide are 1.465 and 1.460, what is the value of d required to ensure single mode performance at a wavelength of 1.50 μm?



QUESTION 10

A circular loop of radius a lies in the $x - y$ plane with the centre of the loop located at the origin. An oscillating current $I = I_0 e^{-i\omega t}$ flows through the loop. A distance $r >> a$ from the origin the electric and magnetic fields are given by

$$\begin{aligned} \vec{H} &= (ka)^2 \, I_0 \frac{e^{i(kr-\omega t)}}{4r} \sin\theta \, \hat{\theta} \\ \vec{E} &= -\,(ka)^2 \, Z_0 I_0 \frac{e^{i(kr-\omega t)}}{4r} \sin\theta \, \hat{\phi} \end{aligned} \tag{1}$$

where $\hat{\theta}$ and $\hat{\phi}$ denote the unit vectors in spherical coordinates, $k = \omega/c$, and $Z_0 = \sqrt{\mu_0/\epsilon_0} \approx 376.7\,\Omega$.

(a) Calculate the complex Poynting vector $\vec{S} = \frac{1}{2}\vec{E} \times \vec{H}^*$, for $r >> a$.

(b) Show that the total (rms) power radiated by the loop can be be written as $P = I_0^2 R_{\text{rad}}$, where R_{rad} denotes the radiation resistance of the coil, and derive the expression for R_{rad}.

(c) If the ohmic resistance of the loop is 1.0Ω calculate the ratio a/λ at which the radiation resistance is equal to the ohmic resistance.