2006 Canadian Association of Physicists University Prize Examination

Tuesday, February 7, 2006

Duration: 3 hours

Instructions:

Calculators are permitted.

There are ten (10) questions; each question has the same value.

Each question will be marked by a different examiner; therefore, ***each question is to be answered on a separate page***. If more than one page is required, the pages for that question should be securely stapled together.

The number of the question, the name of the candidate, and the name of the university/department must be clearly indicated on the first page of ***each*** answer.

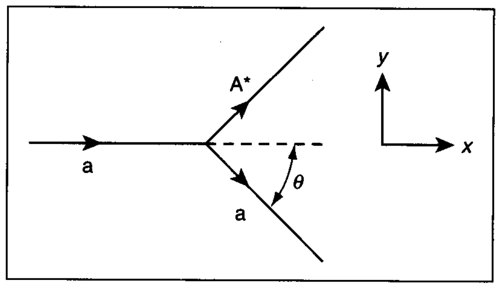
Attempt as many questions as you can, in whole or in part. It is unlikely that you will be able to complete all questions, so work primarily on those which you feel most able to answer.

The examining committee members are:
Carl Adams (St. Francis Xavier), Craig Bennett (Acadia), Malcolm Butler (Saint Mary's), Richard Dunlap (Dalhousie), Kevin Hewitt (Dalhousie), Daniel Labrie (Dalhousie), Andrew Rutenberg (Dalhousie), Michael Robertson (Acadia), David Tindall, (Coordinator, Dalhousie), and Walter Zukauskas (Dalhousie).

1. A. An alpha particle (a) of mass m_a and kinetic energy E_i is incident on a nucleus (A) of mass m_A. The alpha particle scatters inelastically at an angle q as shown in the figure and has a final kinetic energy E_f. The nucleus is left in an excited state of mass m_{A^*} and at an energy ΔE with respect to the ground state such that

$$\Delta E = (m_{A^*} - m_A)c^2$$

Derive an expression for ΔE in terms of m_a, m_{A^*}, E_i, E_f and q.
Note: all energies involved may be treated non-relativistically.



B. **(a)** Natural uranium, as found on Earth, consists of two isotopes in the ratio of ${}^{235}U/{}^{238}U = 7.3\times10^{-3}$. Assuming that these two isotopes existed in equal amounts at the time the Earth was formed, calculate the age of the Earth. The halflives of ${}^{235}U$ and ${}^{238}U$ are 7.14×10^{8} years and 4.49×10^{9} years, respectively.

(b) Terrestrial carbon contains 1 part in 7.69×10^{8} of ${}^{14}C$; the remainder being ${}^{12}C$ (98.9%) and ${}^{13}C$ (1.1%). ${}^{14}C$ decays by β^- decay to the ground state of ${}^{14}N$ with a lifetime of 8267 years. The human body is about 20% carbon. Calculate the approximate activity (in Curies, where 1 Curie is 3.7×10^{10} decays per second) from ${}^{14}C$ for an average person.

2. An electron, mass m, is in a one-dimensional potential well for which the potential is zero along a length a $(0 \le x \le a)$ and infinite outside this length.

(a) Obtain an expression for the energy of the electron when it is in its nth lowest state, where n=1 corresponds to the lowest value of the energy.

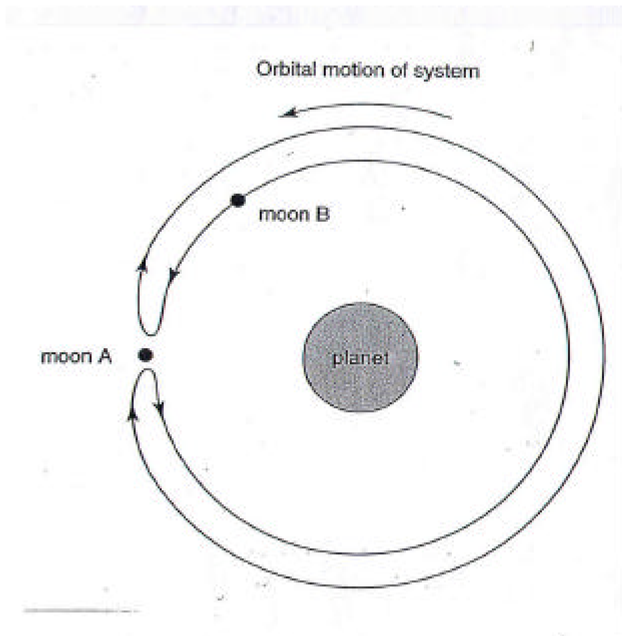
(b) Determine where the electron is most likely to be for n=1 and n=2.

(c) Find an expression for the probability, P, that the electron is within a distance d of x = 0, for arbitrary n.

(d) Interpret the result of part (c) when n becomes very large.

3. **(a)** Two spacecraft are in the same orbit, one ahead of the other; the trailing craft is equipped with thrusters to permit maneuvering. List the steps that must be taken in order for the trailing craft to catch up to the leading craft.
Give reasons for your choices.

(b) The diagram illustrates a so-called "horseshoe orbit". Moons A and B are in circular orbits about the planet, the orbit of moon B being slightly smaller than that of moon A. The horseshoe orbit depicts the movement of moon B as seen in a coordinate system rotating with moon A about the planet. Outline qualitatively how such an apparent orbit might arise. You may assume that the mass of moon B is negligible compared to that of moon A.



(c) A satellite in circular orbit around Earth can alter its orbit by igniting a rocket motor for brief periods. Qualitatively describe how the orbit is changed by firing the rocket in such a way that the satellite is accelerated (i) forward in its orbit, and (ii) radially outward.

(d) The Yarkovsky effect provides a means whereby small asteroids (roughly one metre size) may have their orbits changed. An asteroid absorbs sunlight, and re-emits it as thermal radiation according to Stefan's law. The hottest spot emits the most radiation. Assuming that the asteroid rotates in the same sense that it revolves around the sun, qualitatively describe what will happen to the size of the orbit and its period. How does the situation change if the rotation is in the opposite sense to the revolution?

4. Consider a random walk starting at the origin, R(t=0) = 0, that consists of a randomly oriented step $\vec{r}_i$ of length $|\vec{r}_i| = l$ every time τ. After time t, the displacement is, $\vec{R} = \sum_{i=1,N} \vec{r}_i$ where the number of steps is $N = t / t$.

(a) While the average displacement $< \vec{R} >= 0$, what is $< R^2 >$?

(b) A bacterium swimming at $v = 10mm / s$ randomly changes its direction of motion every time $t = 10s$. What is its diffusion constant D, where $D = \langle R^2 \rangle / (6t)$?

(c) Dead bacteria will still move thermally. Take the bacterium as a sphere of radius $a = 1mm$ with the density of water, what is its root-mean square speed $v_{RMS} = \sqrt{\langle v^2 \rangle}$, when $k_B T = 4 \times 10^{-21} J$?

(d) What is the characteristic time over which a moving bacterium is slowed by viscous drag, with $F_{DRAG} = 6phav$ where the viscosity of water is $h \approx 10^{-3} Nm^2 / s$? Hence, approximately, what is the (Stokes-Einstein) diffusion constant of a dead bacterium?

5. Fermat's principle states that the path of a beam of light minimizes the propagation time.

(a) Use Fermat's principle to derive the law of refraction at a planar interface between two dielectric media with indices n_1 and n_2.

(b) The index of refraction of glass can be increased by diffusing in impurities. It is then possible to make a lens of constant thickness. Given a disk of radius *a* and thickness *d*:

i. Find the radial variation of the index of refraction *n(r)* which will produce a lens with focal length *f* (where *f* >> *a*). Assume a thin lens (*d* << *a*), an index n_o at *r = 0*, and a plane monochromatic wave incident from the left.

ii. Find the point at which rays from the plane wave are brought into focus on the axis.

Note: These graded index refraction (GRIN) lenses are used at the end of optical fibers.

6. A problem you have probably seen in electrostatics involves a charge q that sits a distance d above an infinite, grounded conducting plane. Without loss of generality we can set up coordinates so that the plane coincides with the z=0 (i.e. x-y) plane and the charge sits on the z-axis at (0, 0, d). The problem is solved by considering the extra contribution to the potential from an image charge q' at location (0, 0, $-d'$). In this case you can make the z=0 surface an equipotential with V=0 by setting $q' = -q$ and $d' = d$. The key to this method is that the potential from the image charge allows you to reshape the equipotential surface to match the boundary but does not affect Poisson's equation as long as the image charge remains outside of the region of interest (in this case above the conducting plane).

(a) Is the image charge really there? If not, where are the charges that give the new potential?

(b) Write the potential function that comes from the real charge, q at $z=d$ and image charge q' at $z= -d'$ as described above. Use whatever coordinate system you wish (cylindrical is probably best). Show that, for points in the x-y plane, V=0 (as required for the boundary condition) when $d'=d$ and $q'= -q$.

(c) Now, suppose that the flat conducting plane is curved upward towards the real charge, making a spherical bowl. Explain what happens to the magnitude and position of the image charge. Please include a diagram.

(d) This spherical surface passes through the origin and has a radius of curvature R. For cylindrical coordinates (r, ϕ, z) the equation of the surface is

$$R^2 = (R-z)^2 + r^2 \quad \text{i.e.,} \quad r^2 + z^2 = 2Rz$$

Quantitatively give the values for d' and q' that make this surface a V=0 equipotential. $R > d$ but you shouldn't need to make any approximations.

Hints:

(a) First look at the conditions at the origin to get one simple equation.

(b) If you just look at the boundary condition you will obtain as one solution the trivial answer $q' = -q$ and $d' = -d$, completely canceling the real charge. Although this does satisfy the boundary condition (indeed V=0 everywhere) it fails to satisfy Poisson's equation and is not a solution.

(c) Your answer should revert to the answer for the plane, if R becomes very large.

7. Consider an infinite one-dimensional ionic crystal consisting of a linear chain of N single charged atoms of alternating polarity. Besides the long-range electrostatic interaction, there is a shorter-range repulsive interaction with a potential A/r^3 that acts only between nearest neighbors.

(a) Find the equilibrium atomic spacing for an isolated ionic pair (one atom of each polarity).

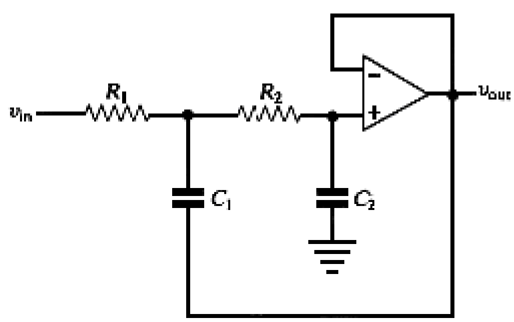
(b) Find an expression for the cohesive energy U(a) of the crystal for atom spacing a.

[hint: $\ln(1+x) = x - (1/2)x^2 + (1/3)x^3 - (1/4)x^4 + \dots$]

(c) Find the equilibrium atom spacing in the crystal, valid for low temperatures.

8. In many electronic applications, active low pass filters with very large roll-off of gain at frequencies beyond the 3dB point are required.

The figure shows the basic building block for such filters; it is a Sallen Key filter, named after its inventors.



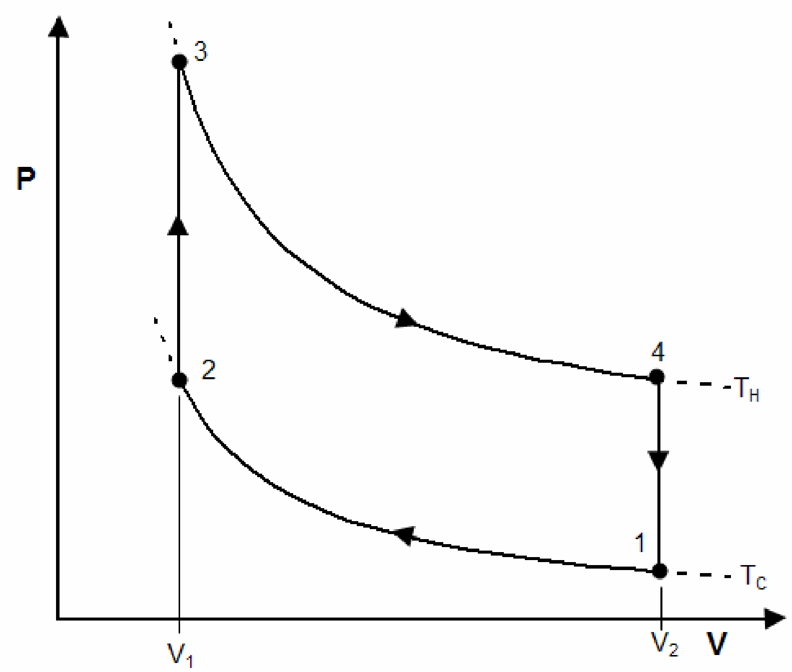
(a) Derive an expression for its gain as a function of frequency, f.

(b) Set $R_1 = R_2 = 1$ kΩ and $C_1 = 2C_2 = 0.20$ μF.

(i) Generate the Bode plot of the magnitude of the gain (in dB), versus log (f).

(ii) Plot the phase angle as a function of log (f).

9. An idealized Stirling engine cycle can be represented by the following P-V diagram. The two isotherms have been indicated by their respective temperatures where H denotes the hotter temperature and C the colder temperature.



Assume that the engine operates in a reversible manner using an ideal gas. In addition, constant heat capacities can be assumed.

(a) Calculate the work for each of the four steps making up the cycle.

(b) Calculate the amount of heat entering or leaving each step of the cycle.

(c) Calculate the change in entropy for each step of the cycle.

(d) What is the total work performed for one complete cycle in terms of the volumes V_1 and V_2 and temperatures T_H and T_C?

(e) If a regenerator is employed in the engine, the heat entering the cycle during the isochoric heating step (2 to 3 in the figure) is supplied internally. In this case, show that the efficiency, h, of the idealized Stirling engine is the same as for a Carnot cycle. That is,

$$h = 1 - \frac{T_C}{T_H}.$$

(f) Calculate the total change in entropy for one cycle of the engine. Does this value make sense? Why?

10. **Answer up to seven (7) of the ten (10) parts of this question:**

(a) Find the entropy per unit volume of black body radiation, given that the energy per unit volume is σT^4, where σ is a constant.

(b) In the sodium atom the 3p states of the valence electron lie higher in energy than the 3s states. Explain this qualitatively.

(c) Draw the electrical circuit which has impedance

$$Z = \frac{1}{jwC} + \frac{1}{\frac{1}{R_1} + \frac{1}{R_2 + jwL}}$$

(d) During the day the sky is blue (sometimes!) Why? Why are clouds not blue?

(e) As seen from Earth, Venus is never more than 46.3° from the Sun. Find its period of revolution around the sun.

(f) Does air friction cause the speed of an earth satellite to increase or decrease? Why?

(g) Why can a ladder be leant at an angle on a rough floor and against a smooth wall, but not on a smooth floor and against a rough wall?

(h) A police officer's radar gun has a diameter of about 100mm. If it operates at a frequency of 10GHz, how wide is the beam at 100m?

(i) Estimate the mass of cold water which could be brought to the boil using the energy dissipated when a car is brought to rest from 100 km h^{-1}.

(j) Estimate the change in the time of sunrise if, during the night, the velocity of light were to halve.