Higher mathematics was created by Newton and Leibniz in the 17th century, and even in their works it was a highly developed discipline. Of course, its development did not stop there. Newton and Leibniz did not confine themselves to the basic definitions and initial theorems. Their contribution was much greater. Newton clearly realized the importance of *differential equations* in analyzing natural phenomena. Take his famous "Principia" (1687), which contains Newtonian mechanics (the Newtonian world system), and you will see it starts with the differential equation of motion (see Eq. (9.4.2)). This equation is taken as an axiom, whereas all subsequent propositions of mechanics are theorems derived from this axiom (and also from the law of gravity that follows from experimental findings (Kepler's laws) and axiom (9.4.2)). In his mathematical investigations Newton formulated "the main problems of mathematical analysis" as follows:

(1) from a given relationship between fluents (initial functions) to determine the relation between fluxions (derivatives);

(2) from a given equation containing fluxions to find the relationship between fluents.

The first of these problems is obviously a problem of the differentiation of known combinations of functions: thus, in Newton's notation, if $z = uv$, then $\dot{z} = \dot{u}v + u\dot{v}$, where u, v, and z are fluents and $\dot{u}$, $\dot{v}$, and $\dot{z}$ are fluxions. The second problem is a problem of the solution of differential equations.

Newton also devoted much attention to infinite *series* (discovery of the binomial series made a great impression on him[6.17]). The whole of Newton's theory of fluxions and fluents (derivatives and integrals) was obviously the fruit of his investigations into the theory of infinite series. Nor is it accidental that the basic theorems (see formulas (6.1.18) and (6.1.19)) of the theory of series are linked with the names of Newton's pupils and associates: Brook ***Taylor*** (1685-1731), who was Secretary to the Royal Society of London (the British academy of sciences) when Newton was its President, and Professor Colin ***Maclaurin*** (1698-1746) of Edinburgh University, who knew Newton personally and greatly admired him. Incidentally, both formulas, (6.1.18) and (6.1.19), were first recorded by Taylor. Maclaurin's contribution was that he posed the question of the sphere of applicability of the formulas and partially answered it. The exact formula (6.1.16), the one containing a finite number of terms, was established by Lagrange (we will return to him again later on).

Leibniz, too, made important contributions to the theory of series and to differential equations in the solution of various geometrical problems. Note that Leibniz was more interested in geometry than in mechanics: he associated the concept of the derivative with a tangent and not speed. Significantly, his fundamental memoir on the "new mathematics" had an involved title: "A New Method of Maxima and Minima, and Also of Tangents, for Which Method Neither Fractions Nor Irrational Quantities are an Obstacle, and a Special Kind of Calculus for This".

The excellent school of Leibniz, headed by the two brothers, ***Jakob Bernoulli*** (1654-1705) and ***Johann Bernoulli*** (1667-1748), made an outstanding contribution to the differential and integral calculus. They were the founders of the famous Swiss family of mathematicians, which in the 17th and 18th centuries produced eight first-class mathematicians. (The most prominent of the later members of the family was Johann's son ***Daniel Bernoulli*** (1700-1782), who worked in Basel and St. Petersburg and was one of the founders of hydrodynamics and the kinetic theory of gases. We will have occasion again to discuss his work.)

6.17 For an account of how this outstanding discovery was evidently made, see, for example, G. Polya, *Mathematical Discovery (On Understanding, Learning and Teaching Problem Solving)*, Vol. 1, Wiley, New York, 1962, pp. 91-93.

Jakob Bernoulli

Johann Bernoulli

Jakob Bernoulli received a theological education, but his interest in mathematics prevailed. He studied the mathematical literature by himself and came upon the works of Leibniz, which made a profound impression on him. In fact, they so overwhelmed him that he gave up the secure life of a pastor and for a number of years held the little-respected and low-paid post of a home tutor. It was only thanks to the good offices of Leibniz that in 1683 he was appointed a professor (at first, of physics, and later of mathematics, too) at the University of Basel.

Johann Bernoulli's father wanted him to go into commerce, but thanks to his elder brother, who had taught him mathematics and physics, his clearly expressed scientific interests ran counter to his father's wishes. Since Switzerland in those years provided very little opportunity for the study of mathematics, Johann Bernoulli was compelled to obtain a medical education and for many years earned his living as a physician. His doctoral dissertation on the movement of muscles was interesting in that it was evidently the first attempt to apply the methods of higher mathematics to physiology. On the recommendation of the famous Christian Huygens, Johann was appointed to a professorship in physics at Groningen (the Netherlands) in 1695. Returning to Basel in 1705, he at first could obtain a university post only as a professor of Greek. It was only after the death of his brother Jakob that Johann Bernoulli was appointed professor of mathematics at the University of Basel and held this post to the end of his life.

The Bernoulli brothers maintained a lively correspondence with Leibniz, who time and again expressed his admiration for their successes. It was precisely in the course of this correspondence that mathematical analysis received its present form, basic symbolism, and terminology. For example, originally Leibniz had been inclined to speak of "differential calculus" and "summational calculus" as the two branches of the "new mathematics," but on a suggestion from Johann Bernoulli he finally chose the Latinized term "integral calculus" (instead of "summational"). Jakob and Johann Bernoulli greatly advanced the new calculus, obtaining, in particular, important results in the theory of differential equations (see Section 6.6) and laying the foundations of what is called *calculus of variations*, which is mentioned in passing in Section 7.2 (see also the text above Example 5 in Section 7.1).[6.18]

The first printed textbook of differential and integral calculus, entitled "Infinitesimal Analysis for the Study of Curves" (note how it echoes the famous memoirs of Leibniz) appeared in 1696. Its author was Guillaume F. A. ***L'Hospital*** (1661-1704), a French nobleman (Marquis de St. Mesme) who was a pupil of Leibniz and Johann Bernoulli. This excellent textbook went through many editions and was translated into many languages. The ideas L'Hospital set forth closely follow the lectures which he heard Johann Bernoulli deliver in Paris; they also follow Bernoulli's manuscript manual "Lectures on Differential Calculus," which was discovered in the library of the University of Basel in the 20th century and was published for the first time in 1922. Thus, while Bernoulli's text remained unknown, the lectures which were based on it and were attended by only one student,

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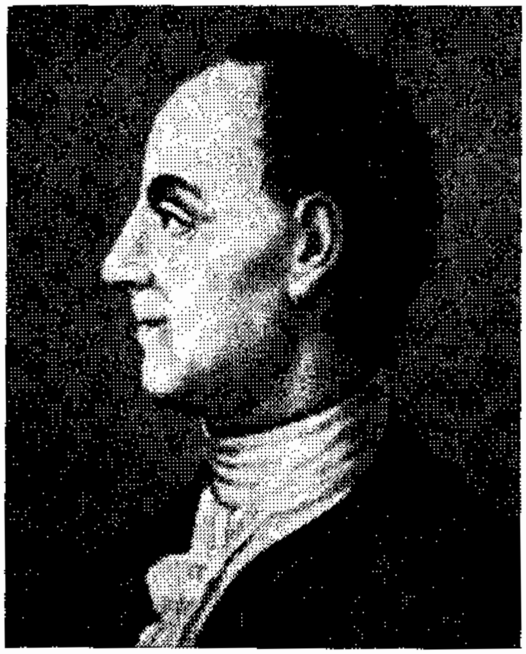
[6.18] Variational calculus became an independent scientific discipline in the 18th century thanks to the works of Euler and Lagrange.



Leonard Euler

Jean Le Rond d'Alembert

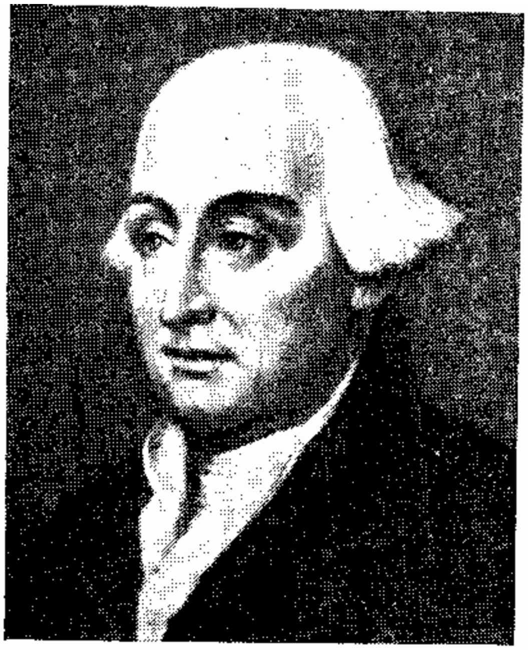
L'Hospital, exerted a tremendous influence on the entire subsequent development of higher mathematics (in particular, it spread the symbols and terminology of Leibniz). Among other things, L'Hospital's textbook saw the first publication of the method of calculating limits which Johann Bernoulli taught his pupil and which is now with so little justification simply called "L'Hospital's rule" (see Section 6.5).

Another of Johann Bernoulli's pupils was the Swiss mathematician Leonhard ***Euler*** (1707-1783). He was introduced to Bernoulli by his father, Pastor Paul Euler, who had once studied mathematics under Jakob Bernoulli. Pastor Euler wanted his son to be a pastor too, but Johann Bernoulli convinced Paul Euler that the boy had outstanding mathematical ability. On Johann Bernoulli's recommendation Leonhard Euler went to St. Petersburg in 1727, where an Academy of Sciences had recently been founded and where two of his teacher's sons (one of them was Daniel Bernoulli) were working. It was originally planned that Euler would take the vacant post of Professor of Physiology at the St. Petersburg Academy, and he made a thorough study of this science in order to follow his teacher's example and apply mathematical methods in it. But when he arrived in St. Petersburg, he found that the post of professor of mathematics was also vacant. So he gave up physiology and devoted himself to mathematics, physics, mechanics, and astronomy (for one, the movements of the moon).

Leonhard Euler spent a large part of his life in St. Petersburg. There was a break of 25 years when, on invitation from King Frederick II of Prussia, he moved from St. Petersburg, in those years a place not very conducive to calm scientific research, to Berlin.[6.19] Here he became head of the physics and mathematics department of the Berlin (Prussian) Academy of Sciences, but he kept up his close contacts with the St. Petersburg Academy. Euler was the most prominent and most productive mathematician of the 18th century. His work dealt with literally every sphere of mathematics and mathematical physics. We write about some of Euler's findings below (see Section 10.8 and Chapters 14 and 15). Euler's multivolume textbook of differential and integral calculus was translated into other languages.

Among Euler's contemporaries was Jean Le Rond ***d'Alembert*** (1717-1783), a distinguished French mathematician and author of treatises on mechanics. His exceptional scientific versatility is particularly evident in his collaboration with Denis ***Diderot*** (1713-1784). They produced the famous "Encyclopédie, ou Dictionaire Raisonné des Sciences, des Arts et des Métiers," in which he wrote nearly all the articles dealing with the natural sciences. D'Alembert was named after the church (Jean Le Rond; literally, Jean the Round) on the steps of which he was found as an abandoned infant. Notwithstanding this handicap he became a prominent man of science. In Section 10.8 we describe a most edifying and fruitful scientific debate in which Leonhard Euler, d'Alembert, and Daniel Bernoulli took part. Both Euler and d'Alembert strove for concrete

[6.19] This refers to the "Biron period" in Russian history, named after E. J. Biron, a disreputable favorite of Empress Anne (1693-1740), whom she brought from Courland. The reign of Empress Anne, from 1730 to 1740, was marked by arbitrary arrests and executions of innocent people and total disorganization of the machinery of government.



Joseph Louis Lagrange

Augustin Louis Cauchy

results, displaying outstanding intuition in mathematics and physics and a lack of interest in scholastic discussions of mathematical ideas. This is illustrated, for example, by d'Alembert's well-known call to the young: "Keep on working—complete understanding will come in time."

When Euler decided to return to St. Petersburg after having headed the physico-mathematical department of the Berlin Academy from 1741 to 1766, the question of his successor arose. Euler recommended the young mathematician Joseph Louis ***Lagrange*** (1736-1813), who was born into a French family that had moved to Italy. Lagrange was only 30 at the time. His candidature was enthusiastically supported by d'Alembert, who corresponded with King Frederick II. Lagrange was already a well-known scientist. He had begun to teach in the Turin Artillery School in 1726, when he was 20, and the following year he was one of the founding members of the Turin Academy of Sciences, in whose proceedings he published many papers. In 1787, after the death of Frederick II, Lagrange moved to France, where he played an outstanding role in the rise of the Parisian Polytechnical School (École Polytechnique), an institution of higher learning of a new type, which trained research engineers.[6.20] It was during the Parisian period that Lagrange compiled his two-volume "Analytical Mechanics" (1788), which greatly furthered mathematics and physics. Lagrange published an excellent, and in many respects revolutionary, textbook of mathematical analysis ("The Theory of Analytic Functions," 1797) based on lectures that he delivered at the Polytechnical School. He is also known for his fundamental studies in algebra. In the breadth and range of his mathematical interests and in the brilliance of his achievements, Lagrange was second perhaps only to Euler among the mathematicians of the 18th century.

One of the pupils and ardent admirers of Lagrange was Jean Baptiste ***Fourier*** (1768-1830), a professor at the Polytechnical School whose name is linked with outstanding achievements in the field of *partial differential equations* (also called *equations of mathematical physics*; see Section 10.8), for one, in the theory of the propagation of heat and in the theory of trigonometric series (see Section 10.9).

Another professor of the Polytechnical School was the famous Augustin Louis ***Cauchy*** (1789-1857), who created the modern theory of limits (incidentally, his main ideas were borrowed from d'Alembert), which helped to clarify all the concepts of higher mathematics. Cauchy is rightly regarded as the father of the *theory of functions of a complex variable* (see Chapters 14, 15, and 17). He shares this honor, incidentally, with Georg Friedrich Bernhard ***Riemann*** (1826-1866), the German mathematician and one of the most prominent scientists of the 19th century, who made outstanding contributions in literally every field of mathematics and mathematical physics.

While Cauchy was a representative of the Polytechnical School of Paris, Riemann was connected with another educational and scien-

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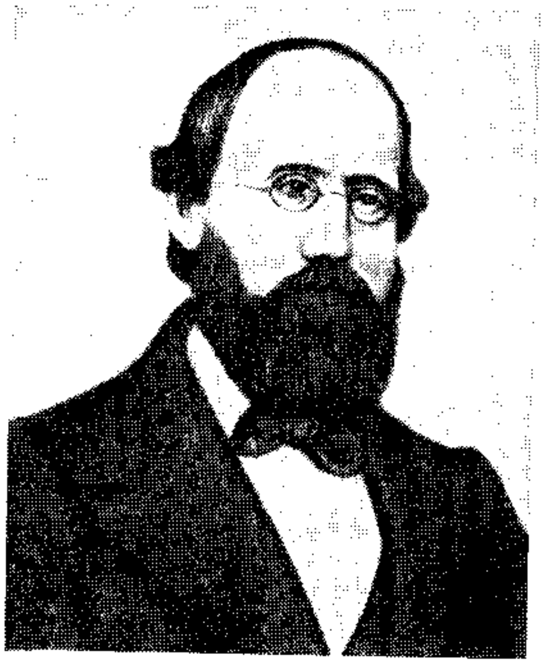
[6.20] The example of the famous Polytechnical School of Gaspard ***Monge*** (1746-1818) and Lagrange was undoubtedly taken into account by the founders of the Moscow Physico-Technical Institute in the Soviet Union and the founders of the Massachusetts and California institutes of technology in the United States.



Georg Riemann

tific center that played a big role in the development of mathematics and physics in the 19th and 20th centuries—the University of Göttingen in Germany. Here he attended lectures by the famous mathematician, physicist, astronomer, and geodesist Karl Friedrich ***Gauss*** (1777-1855), who is usually acknowledged to be the first mathematician of the 19th century. At the University of Göttingen Riemann presented the first-ever course in the theory of functions of a complex variable, as well as a course in the theory of partial differential equations. His lectures in this second course, published after his death, were the first textbook on the equations of mathematical physics.

David ***Hilbert*** (1862-1943) belongs to an altogether different generation of Göttingen scientists. He is regarded as the father of *functional analysis*, which deals with the functions of functions, or functionals (see the text preceding Example 5 in Section 7.1). Today this branch of mathematics includes generalized functions (or distributions), such as Paul Dirac's delta function (see Chapters 16 and 17).

Today the ideas of mathematical analysis continue to undergo intensive development, and computers are giving them a new and powerful impetus.