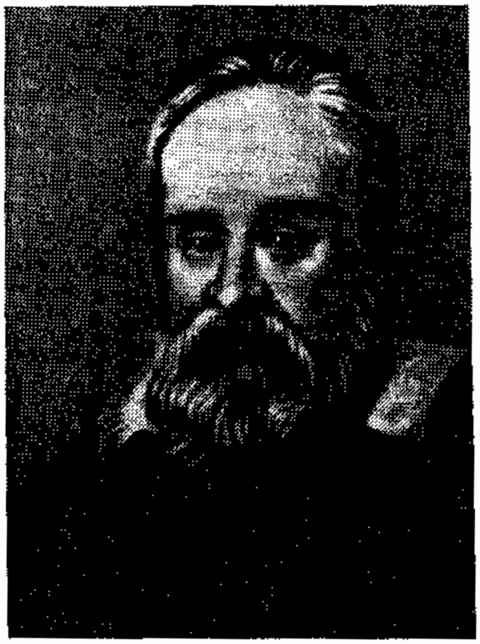
The rise of "higher mathematics", that is, *differential and integral calculus*, was a turning point in the history of civilization: it gave man a powerful tool for analyzing processes of many different kinds, for developing a fundamental explanation of physical phenomena, and for constructing a scientific picture of the world. Actually, it was the thinkers of ancient Greece, primarily the genius of ***Archimedes*** of Syracuse (287-212 B.C.), who skillfully solved the first problems of differential and integral calculus. Archimedes successfully applied mathematics in designing machines and devices (the weapons of war which he invented struck terror into the Romans laying siege to his native Syracuse). Archimedes knew, for example, how to draw tangents to curves and how to calculate an area bounded by curves. He solved the problem of drawing a tangent to what came to be called the ***Archimedes spiral***, that is, the line described by a snail crawling steadily along the spoke of a wheel turning at a constant rate, and also the problem of ***squaring the parabola***, that is, finding the area of a segment of a parabola. Today problems of this kind can be solved without difficulty by any college student (or even a student of higher school), but in those remote times they were within the powers of only a giant like Archimedes. There was no general method for solving such problems. Each one required great effort.

For that matter, the primitive level of technology in ancient Greece did not require the solution of problems that called for great inventiveness but were devoid of applications outside mathematics: as a rule, the thinkers of ancient times regarded mathematics not as an effective method of solving practical problems but only as a theoretical science whose perfection reflected the profound harmony of the world, yet only explained the world in a purely philosophical sense. (Archimedes was practically the only exception in this respect because he closely linked mathematics with mechanics and physics. But he was a genius; in this, as in many other respects, he was far ahead of his time.)

The static nature of life in ancient Greece, where hardly any machines were known and life centered around city squares, palaces, and temples decorated with beautiful statues as immobile as the temples and city squares themselves, gave rise to the *metaphysical thinking* (rigidity) that was so characteristic of the mathematics of antiquity: it was not customary to regard processes in flux. They limited themselves to unchanging "states," a



Galileo Galilei

Johann Kepler

reflection of which were, for the famous geometer ***Euclid*** of Alexandria (early third century B.C.), merely chains of congruent triangles, which occurred frequently in his arguments.

With the flowering of Italian cities in the 15th and 16th centuries and the appearance of the first factories, which heralded the imminent rise of machine-based production, the static metaphysical thinking of the ancients became inconceivable. In this period it could only hinder much needed scientific progress. The great ***Galileo Galilei*** (1564-1642) was the first to proclaim publicly that mathematics provided the key to the mysteries of the Universe (see footnote 3.15). Under Galileo's influence, his pupils—Evangelista ***Torricelli*** (1608-1647) who discovered the principle of the barometer, and the geometrician Bonaventura ***Cavalieri*** (1598-1647) who continued the work of Archimedes, for whom their teacher had such great affection—solved many specific problems which today lie within the province of higher mathematics. For one, Cavalieri evolved a method of calculating volumes that is very similar to those described above. Galileo was the first to see that the problem *of determining the path of an object from its velocity* practically coincides with the problem that had so interested Archimedes, that of *determining the area of curved figures.* Continuing Galileo's work, Torricelli established that the inverse problem, that of *finding the velocity from the path*, was akin to the problem of *drawing a tangent to a curve.* But at that time there did not yet exist general methods of solving such problems; there was no common algorithm to enable one "to calculate without thinking."

Nor did Johann ***Kepler*** (1571-1630), that outstanding astronomer and mathematician whose discoveries played such a big role in building a scientific picture of the world, have any such method. Kepler was indisputably the leading master of integration in his day (though it was not called that yet). In 1614, when Kepler married a second time, he had occasion to buy a good deal of wine for the wedding reception, and he saw how difficult it was to estimate the cubical contents of wine casks of different shapes from the base radius and the height. The problem intrigued him. The following year, 1615, he published his *Nova Stereometria Doliorum Vinariorum* (The New Science of Measuring Volumes of Wine Casks), "with addenda to Archimedean stereometry," as he also informed the reader. Here he collected a large number of problems in determining the volume of bodies limited by curved surfaces and displayed great ingenuity in developing formulas for such volumes, which are today established by integral calculus (see Section 7.10).

René ***Descartes*** (1596-1650) was truly the forerunner of higher mathematics. Soldier, diplomat, natural scientist, and abstract thinker, he was the father of *analytic geometry* (the method of coordinates in geometry; see Chapter 1) and a profound philosopher who declared that the world was knowable. He stated the basic principles of dialectics and the place occupied in life by various processes whose study, he believed, constituted the chief aim of mathematics. Descartes substantially improved the symbols and language of mathematics, giving them their present-day appearance. This greatly stimulated further progress and democratization of mathematical knowledge.

Another French scientist, Pierre ***Fermat*** (1601-1665), a lawyer by profession and a pro-

René Descartes

Pierre Fermat

found amateur mathematician, was a contemporary and, in a way, a rival of Descartes. Working independently of Descartes, Fermat somewhat earlier elaborated (though it was published later) a system of using, in geometry, the methods of coordinates and algebraic computations now known as analytic geometry. Fermat's exposition of the subject was perhaps closer to the modern one than that of Descartes; and the fact that the writings of Descartes have come to hold a much more significant place in the history of science is connected primarily with his active teaching and his more advanced system of notation. (The situation was much the same in the development of differential and integral calculus.) Simultaneously, Fermat occupied himself with problems that are now part of mathematical analysis. He evolved the concept of the *differential* (see Section 4.1 below) and the overall idea that the *maxima and minima of a (smooth) function must lie at the points where the (first) derivative of the function vanishes*, and also carried out ingenious calculations of the values of some integrals (e.g., the method of deducing formula (5.2.1) indicated in Exercise 3.2.3). Descartes, too, made contributions to the field of analysis.

Christian ***Huygens*** (1629-1695) of the Netherlands (a junior contemporary of Descartes and Fermat) created the wave theory of light and was noted for his work in the application of mathematics to mechanics and physics (e.g., he produced the strict mathematical theory of pendulum clocks (see Section 7.9)). In his investigations, however, Huygens used the archaic methods of thinkers of antiquity, Archimedes for one, because he believed that a newer method would yield no advantages inasmuch as he could solve any problem "in the old way." (True, but you had to have the brains of Huygens to do that.)

Two outstanding thinkers of the 17th century, the Englishman Sir Isaac ***Newton*** (1643-1727) and the German Gottfried ***Leibniz*** (1646-1716), are rightly regarded as the real founders of higher mathematics. Both indisputably rank among the most profound scientists that the world has ever known. To them we owe a coherent exposition of the new calculus, a chain of formulas for finding the derivative of any given algebraic function without any difficulty and a complete understanding of the connection between the derivative and the integral and the significance of that connection, which provides a general algorithm for calculating integrals by turning to a list of derivatives (see Chapters 4 and 5).

Leibniz was a truly versatile scientist: he delved into philosophy, philology, history, psychology (in which he was one of the pioneers of penetration into the sphere of the subconscious), biology (he was one of the precursors of the theory of evolution), geology, mining, mathematics, and mechanics (he evolved the concept of "living force," that is, kinetic energy). At the same time he was active in politics and diplomacy (he strove to reconcile the German principalities because he foresaw their eventual union into a single state, and dreamed of uniting the Catholic and Protestant churches). He was also an organizer of scientific academies; for one, he was the founder and first president of the Prussian Academy of Sciences in Berlin, and he had several talks with Peter the Great of Russia about founding a Russian academy of sciences, which was later established in full conformity with his suggestions and wishes. The mathematical

Christian Huygens

Isaac Newton

analysis of Leibniz [to whom we owe, among other things, the modern terms "derivative" (*Ableitung* in German) and "integral,"[3.19] was of a form much like that adopted in our book. Leibniz designated the (first) derivative of a function $y = f(x)$ by the symbol dy/dx; he understood the "differentials" dy and dx as the "limiting" values of the increments Δy and Δx at which we arrive by an unbounded reduction of Δx (neither the word "limit" nor its concept existed in the mathematics of Leibniz). This approach corresponded to introducing the concept of the derivative as the tangent of the angle formed by the tangent line to the graph of the function and the x axis: to a small Δx there corresponds a small triangle MNP (see Figure 7.9.1), where $\tan \alpha_1 = NP/PM = \Delta y/\Delta x$; when Δx is reduced without limit, the increments Δx and Δy are replaced by the differentials dx and dy, and the secant MN of length ds (the differential of the arc length; see Section 7.9) is replaced by the appropriate tangent line (Leibniz called such a triangle the *characteristic* triangle).

Leibniz emphasized in every possible way the algorithmic aspect of the new calculus and the system of rules that automatically guarantee a correct result when seeking derivatives. He also worked out a method of handling differentials (dealt with here in Chapter 4); use of this method provides a recipe for calculating derivatives. Leibniz used the modern symbol $\int f(x)\,dx$ to designate the integral of the function $y = f(x)$.

Leibniz was a tireless teacher, and that plus his felicitous system of notation and terms resulted in the universal acceptance of higher mathematics in the form he developed it. Leibniz is also considered a classic in the field of philosophy (here he did much to extend and to amplify the work done by Descartes). He also deserves credit for many profound ideas that partially became reality only in the 19th and 20th centuries: e.g., the idea of a "geometrical calculus," from which the modern vector calculus later arose; his attempts to "algorithmize thinking," in which, as Leibniz wrote, the two sides in a controversy would no longer have to conduct lengthy debates inasmuch as one of them could always say to the other, "Well, let us verify which of us is right, let us calculate, my dear sir." The rough outlines of this kind of "propositional calculus," found in the papers of Leibniz, closely resemble the mathematical logic of the 19th and 20th centuries.

Newton arrived at the same ideas as Leibniz quite independently and even somewhat earlier. He was perhaps more of a physicist and astronomer than a mathematician; his contributions to the birth of physics and to the rise of a new method in natural science cannot be exaggerated. Joseph Louis ***Lagrange*** (1736-1813), the distinguished 18th and 19th century mathematician and investigator in the field of mechanics (we will discuss his work later on) once said of Newton: "He is the most fortunate of men, for the system of the world can be discovered only once." The

[3.19] At first Leibniz simply spoke of the sum (and also "summational calculus" instead of "integral calculus"); he enthusiastically adopted the terms *integral* (from the Latin *integer*, meaning whole) and *integral calculus* proposed by his pupils, the brothers Jakob and Johann Bernoulli.

Gottfried Leibniz

same idea is conveyed in the famous "Epitaph for Sir Isaac Newton" written by his contemporary Alexander ***Pope*** (1688-1744):

> Nature, and Nature's laws lay hid in night: God said, *Let Newton be!* and all was light.

Newton, of course, identified the derivative with velocity: he regarded its properties as the physical properties of velocity. Yet the formal mathematical theory of derivatives (and also integrals), founded on a variation of the theory of limits, was not alien to him either. However, in the absence of a definition of the term limit, the theory did not make his constructions more flawless than the (logically not indisputable) manipulations of Leibniz with infinitesimals, but only made them more unwieldy; also the fact that Leibniz clearly exerted a greater influence on European mathematics than Newton may be connected with this.

Newton gave the name of *fluxion* to the derivative, and the initial function from which the derivative was calculated was called the *fluent* (from the Latin *fluere*, to flow), emphasizing that the quantities under consideration were variable; the fluxion arose as the rate of change of the fluent, while the fluent was restored from the fluxion as the path from the speed. Newton began his exposition of the analysis with two basic problems, to which all the others are reduced.

1. Knowing the length of the path traversed, find the speed of motion over a fixed time interval.

2. Knowing the speed of motion, find the length of the path traversed over a fixed time interval.

These are obviously the problems of finding the derivative from the given function, and of finding the function from its derivative, that is, the problem of calculating an indefinite integral. The fact that these two problems (in their geometrical presentation, the drawing of a tangent to a given curve and the calculation of the area bounded by the curve) are inverses of each other had evidently been discovered first by Newton's teacher, Isaac ***Barrow*** (1630-1677), who subsequently resigned his post as Lucasian Professor of Mathematics at Cambridge (a rare case in the history of science!) in favor of his brilliant pupil because he felt that Newton was more worthy of the post than he. However, it is not at all by accident that the connection between derivatives and integrals is named after Newton and Leibniz instead of after Barrow, for only these two great scientists realized the full extent of the mutually inverse nature of the operations of differentiation and integration, which had been discovered by Barrow (and also, independently, by Leibniz), and the possibility of making this fact the foundation for a broad calculation of derivatives and integrals. It is to the far-sightedness of Newton and Leibniz that mathematical analysis owes the rapid progress that began immediately after the first publications and statements by these two great scientists and which truly marked the dawn of a new era, the era of great scientific discoveries, the era of a sweeping assault on Nature's secrets to promote human welfare.

Priority in applying differential and integral calculus to fathom Nature's secrets undoubtedly goes to Isaac Newton, who put forward the general idea that the laws of Nature must have the form of *differential equations* linking the functions that describe the phenomenon under study. We owe to Newton the formulation of the basic equations of motion (for more details see Chapters 9 and 10), which he then applied to deduce the law of gravitation and to study the motion of celestial bodies and the fairly complicated problems of celestial mechanics (see Lagrange's above-cited assessment of Newton). In effect, Newton's work gave birth to a theoretical physics based on the use of a profound mathematical apparatus, which physics, in its substantive part, has gone far beyond the limits of the problems which Newton grappled with and, in its mathematical part, rests on the whole body of modern science, many times surpassing the differential and integral calculus created by Newton and Leibniz.

Newton designated a fluxion (derivative) by a dot placed above the letter symbolizing the initial function. For example, he wrote the derivative of the function $y = f(x)$ as $\dot{y}$. (This notation is retained today in mechanics, but we will not use it.) Newton proposed designating the operation of a transition from a fluxion to a fluent by a dot placed underneath:

thus, if $y = x^2$ and $y_1 = 2x$, then, in this system of notation, $\dot{y} = y_1$ and $\dot{y_1} = y$. The symmetry of these designations makes them attractive. The theorem of the connection between the derivative and the integral can be written as follows: $\dot{y} = y$, where the expression $\dot{y}$ can be understood in two ways: as the fluxion of the function y and as the fluent of the function $\dot{y}$. But today this appealing notation has only historical significance (besides, Newton himself was not particularly consistent in using it). The designation y' for the derivative of a function was introduced by the French mathematician Augustin ***Cauchy*** (1789-1857), whom we will be hearing more about later.

One of the regrettable episodes in the history of science was the bitter controversy between Newton and Leibniz, conducted chiefly by their admirers and pupils and not by the two outstanding scientists themselves.[3.20] It was a controversy that brought neither honour nor advantage to either side. Leibniz, accused (quite groundlessly, as we know today) of directly borrowing his ideas from Newton, lost the patronage of the dukes of Hanover, who had long supported him, and died in poverty and oblivion. His death was not reported in a single German (to say nothing of an English) publication; and even the Berlin (Prussian) Academy, which he founded, took no notice of his death. Only the French Academy, of which Leibniz had been an active member, paid tribute to him in a eulogy. On the other hand, the refusal to recognize Leibniz (and also the refusal to recognize his differential and integral calculus) substantially hindered the progress of English science and completely separated English pure and applied mathematics from continental mathematics: English university graduates were not familiar with the terms "derivative" and "integral" or with the notation introduced by Leibniz, and hence were unable to read books and treatises by German or French scientists. It seems to us, however, that the disagreement between Newton and Leibniz was not accidental and deserves a more detailed examination of its causes.

The fact that Newton and Leibniz simultaneously and, beyond dispute, independently discovered differential and integral calculus best of all demonstrates the timeliness and inevitability of the great scientific revolution of the 17th century. Moreover, the different (even opposite) psychic make-up and scientific programs of these two outstanding scientists, which shaped the pathways by which they arrived at their discoveries, make a comparison of their investigations highly edifying.[3.21] Leibniz, the philosopher, was largely guided by what he called the "metaphysics of infinitesimals" (differentials; incidentally, this concept, which came from Fermat, appeared in Newton's works under another name before it did in those of Leibniz). Leibniz was carried onwards by the language he had created, by the developed symbols and terminology characteristic of the new calculus. (We remind our readers once again of Leibniz's great interest in the science of language: he was one of the founders of what is now known as comparative-historical linguistics.)

Moreover, the definition (later included in all textbooks) of infinitesimals as variables whose limit is equal to zero was undoubtedly alien to Leibniz. It corresponded much more closely to the scientific thinking of Newton. Leibniz regarded infinitesimals as "special numbers," the rules for using which completely differed from those governing operations car-

[3.20] The Royal Society of London appointed a special committee to discuss the controversy over priority (the committee sided with Newton). This committee, naturally, did not include scientists whose works it discussed. More than that, the first phrase of the committee's report (published under the title Commercium Epistolicium in 1712) stated that only the absence of the scientists involved in the controversy could ensure impartiality. But, alas, a copy of the draft of the report written in the hand so well known to historians of science shows with all certainty that the report (including the first phrase) was written completely by the then President of the Royal Society, Sir Isaac Newton. On the other hand, the behavior of Leibniz in the controversy with Newton can also hardly be considered irreproachable.

[3.21] These differences in approach and viewpoint make, we believe, the whole controversy over priority quite meaningless, since now it can be said that the inventions of Newton and Leibniz were entirely different (e.g., see A.R. Hall, *Philosophers at War: The Quarrel Between Newton and Leibniz*, Cambridge University Press, Cambridge, 1980).

Note that analytic geometry was created simultaneously and independently by Descartes and Fermat; moreover, these two scientists also belonged to different psychological types. In contrast to Descartes, who thought largely in terms of physics (and also geometry), Fermat can be regarded as an algorithmist similar to Leibniz. This is reflected in his far more systematic treatment of the subject of analytic geometry. (The "algorithmicity" of Fermat's thinking was most fully manifested in his outstanding achievements in the theory of numbers, a theory that did not interest Descartes in the least; incidentally, Fermat had a most profound grasp of physics—he discovered a remarkable principle that the path of a ray of light from one point to another, through one or more media, is such that the time taken is a minimum. Fermat's principle played an outstanding role in the further development of physics.

ried out with ordinary (real) numbers.[3.22] Newton, the physicist, however, proceeded from the substantive meaning of the new concepts and their role in natural science: he regarded the derivative as speed, and the reason why he did not carry through to the end the theory of limits which he had set forth in outline was because he saw no particular need for it. In his view, the physical meaning of all the concepts that he introduced fully justified them, and he felt no need for formal mathematical theories here.

We can assume that the profound mutual antipathy that arose between Newton and Leibniz occurred at the time of their first (and, apparently, only) meeting. This was when Leibniz came to England to demonstrate his version of a "mathematical" (to be more exact, arithmetical) machine [the first to put forward this idea was the outstanding 17th-century French mathematician and physicist Blaise ***Pascal*** (1623-1662)[3.23]: the very idea of such a machine ran counter to Newton's type of intellect]. Still more remote from Newton was Leibniz's dream (and prevision) of "logical machines" (actually, the prototypes of today's computers) for mechanizing mental processes. On the other hand, Leibniz completely rejected Newtonian mechanics (although he had an exceptionally high opinion of Newton as a mathematician), because the very idea of action-at-a-distance (Newtonian gravitation) contradicted his general scientific and religious views.

The subsequent evolution of the scientific ideas of Newton and Leibniz in European science is not without interest. Newton emerged the indisputable victor in the debate about priority, but the outward forms of differential and integral calculus have come down to us entirely from Leibniz. As for scientific ideology in the broad sense of the word, this was influenced for many centuries to a far greater degree by Newton than by Leibniz. This is illustrated not only by the above-cited lines from Pope and Lagrange. The entire history of European science in the 18th, 19th, and first half of the 20th centuries was inspired by Newtonian mechanics as a true model of a scientific theory in the finest meaning of the term. And our book, too, owes much more to the views of Newton than to the world-outlook of Leibniz.

But in the second half of the present century Leibniz's dream of "thinking machines" has suddenly come true. What is more, today's "computerization of knowledge" has once again put the algorithmic ideas of Leibniz in the forefront of scientific thinking, not to mention the significance of diverse particular achievements of the great German, from the elements of *mathematical logic* he created to his serious interest in *combinatorics*, something quite unusual for the 17th century.

Summing up, we can say that while Newton and Leibniz approached higher mathematics from different starting points, the subsequent history of science has confirmed the unquestionable value of *both* avenues of thought which they represent, the value of both Newton's approaches and the scientific ideas of Leibniz.

[3.22] Interestingly, these ideas of Leibniz received a fully logical substantiation only in 1960 in what is known as nonstandard analysis introduced by the American mathematician Abraham ***Robinson*** (1918-1974), which has already found definite application.

[3.23] In some respects Leibniz's machine surpassed Pascal's arithmetical machine: it not only could perform arithmetic operations on numbers but could, for instance, extract square roots.