The Two-body problem

Big simplification: the problem of two bodies interacting via potential between them can be reduced to the problem of a fictitious single body of reduced mass $\mu = \frac{m_1 m_2}{m_1 + m_2}$ interacting with same potential (fixed at a point)

N.B. called reduced mass because always less than m_1 or m_2

Centre of mass and relative coords

$\vec{r}_1, \vec{r}_2, m_1, m_2, \vec{F}$ (force on ~~m~~ m_1 due to m_2), $\vec{g}$

~~N.B. we shall add a uniform gravitational field because problem is no harder to solve, but extends applicability.~~

Eqns of motion

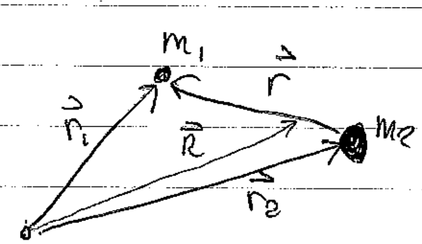
$$m_1 \ddot{\vec{r}}_1 = m_1 \vec{g} + \vec{F}$$

$$m_2 \ddot{\vec{r}}_2 = m_2 \vec{g} - \vec{F}$$

Position of Centre of mass (C.O.M)

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} \qquad M = m_1 + m_2$$

Relative position $\vec{r} = \vec{r}_1 - \vec{r}_2$



Get eqn of Motion for $\vec{R}$ by adding: $M \ddot{\vec{R}} = M \vec{g}$

" " " " " $\vec{r}$ by ÷ by masses and subtract: $\ddot{\vec{r}} = \frac{\vec{F}}{m_1} + \frac{\vec{F}}{m_2} = \vec{F}\left(\frac{m_2 + m_1}{m_1 m_2}\right) = \vec{F}/\mu$

~~where~~ ~~..~~ COM and relative eqns of motion are INDEPENDENT.

COM moves with const. acceleration g, and if $g=0$

$$M\dot{\vec{R}} = \vec{P} = \text{const}$$

Relative eqn of motion is identical to that of single particle of mass μ moving under influence of $\vec{F}$

After calc. can go back:

$$\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r} \qquad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

(e.g. sub $r_1 = \vec{r} + \vec{r}_2$ into $\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}$)

How about $\vec{J}$?

$$\vec{J} = m_1\vec{r}_1 \wedge \dot{\vec{r}}_1 + m_2 \vec{r}_2 \wedge \dot{\vec{r}}_2$$

$$= m_1\left(\vec{R} + \frac{m_2}{M}\vec{r}\right)\wedge\left(\dot{\vec{R}} + \frac{m_2}{M}\dot{\vec{r}}\right) + m_2\left(\vec{R} - \frac{m_1}{M}\vec{r}\right)\wedge\left(\dot{\vec{R}} - \frac{m_1}{M}\dot{\vec{r}}\right)$$

$$= m_1\vec{R}\wedge\dot{\vec{R}} + \frac{m_1 m_2^2}{M^2}\vec{r}\wedge\dot{\vec{r}} + \frac{m_1 m_2}{M}\vec{R}\wedge\dot{\vec{r}} + \frac{m_1 m_2}{M}\vec{r}\wedge\dot{\vec{R}}$$

$$+ m_2\vec{R}\wedge\dot{\vec{R}} + \frac{m_2 m_1^2}{M^2}\hat{r}\wedge\dot{\vec{r}} - \frac{m_2 m_1}{M}\vec{r}\wedge\dot{\vec{R}} - \frac{m_2 m_1}{M}\vec{R}\wedge\dot{\vec{r}}$$

cross terms cancel

$$= M\vec{R}\wedge\dot{\vec{R}} + \underbrace{\frac{m_1 m_2(m_1+m_2)}{M^2}}_{=\mu}\vec{r}\wedge\dot{\vec{r}}$$

Also $T = \frac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\vec{r}}_2^{\,2} = \frac{1}{2}M\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2$

<u>COM frame</u> : COM is at rest at origin
(if $g=0$, otherwise it is a uniformly accelerated frame)

We can choose in COM frame : $\vec{R}^* = 0$ ← COM frame (r⃗ is independent of choice of origin)

then $\vec{r}_1^* = \frac{m_2}{M}\vec{r}$, $\vec{r}_2^* = -\frac{m_1}{M}\vec{r}$

Momenta are EQUAL and OPPOSITE !

$$m_1 \dot{\vec{r}}_1^* = -m_2\dot{\vec{r}}_2^* = \mu\dot{\vec{r}} = \vec{p}^*$$

~~Ba~~ Solve problem in COM frame then go back:

$$\dot{\vec{r}}_1 = \dot{\vec{R}} + \dot{\vec{r}}_1^* \qquad \dot{\vec{r}}_2 = \dot{\vec{R}} + \dot{\vec{r}}_2^*$$

momenta:

$$\vec{p}_1 = m\dot{\vec{r}}_1 = m_1\dot{\vec{R}} + p^* \qquad \vec{p}_2 = m_2\dot{\vec{r}}_2 = m_2\dot{R} - \vec{p}^*$$

Easy to show:

$$\vec{P}^* = 0$$
$$\vec{J}^* = \mu\vec{r}\wedge\dot{\vec{r}} = \vec{r}\wedge\vec{p}^*$$
$$T^* = \tfrac{1}{2}\mu\dot{r}^2 = \frac{\vec{p}^{*2}}{2\mu}$$

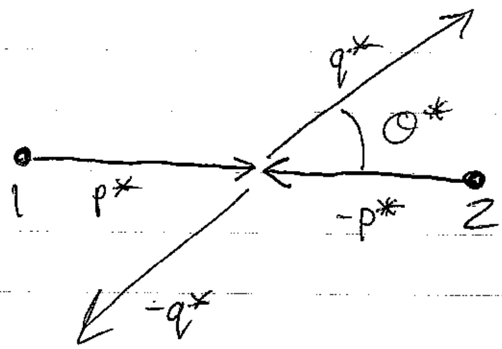
So in an other frame

$$\left.\begin{aligned}\vec{P} &= M\dot{\vec{R}}\\ \vec{J} &= M\vec{R}\wedge\dot{\vec{R}} + \vec{J}^*\\ T &= \tfrac{1}{2}M\dot{\vec{R}}^2 + \vec{T}^*\end{aligned}\right\}$$

to add contribution of a particle of mass M located at centre of mass $\vec{R}$

Collisions

~~Elastic collision~~ In COM frame



each particle is scattered through same angle θ^*

If Elastic collision : $T^* = \frac{p^{*2}}{2\mu} = \frac{q^{*2}}{2\mu}$

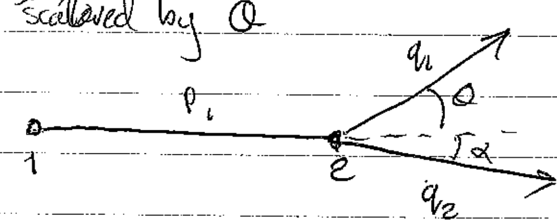
$\Rightarrow \quad p^* = q^*$ (magnitudes same)

Laboratory frame : most experiments have one particle at rest in laboratory ("target") $\Rightarrow \vec{p}_2 = 0$

before $\vec{p}_1$, $\vec{p}_2 = 0$

after $\vec{q}_1$, $\vec{q}_2 \rightarrow$ recoil angle α.

scattered by θ



Calc. is simple in COM frame :

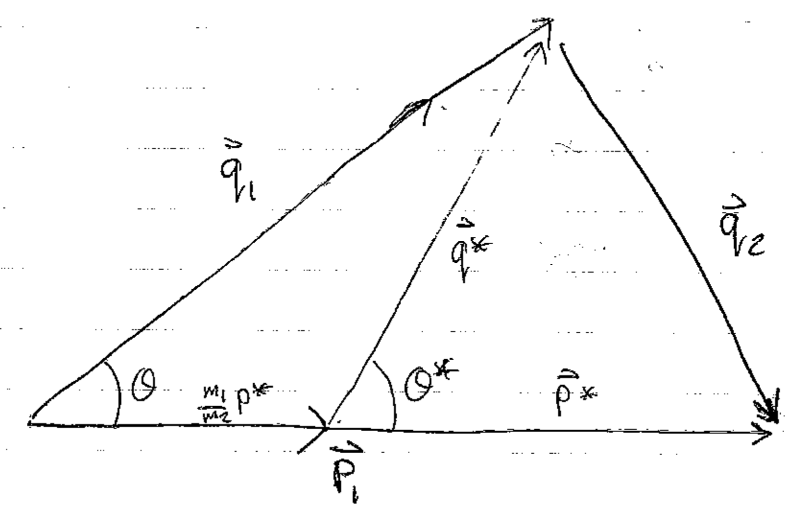
$\vec{p}_2$ is zero so $\dot{\vec{R}} = \frac{1}{m_2}\vec{p}^*$ (from $m_2\dot{\vec{r}}_2 = m_2\dot{\vec{R}} - \vec{p}^*$)

then $\vec{p}_1 = \frac{m_1}{m_2}\vec{p}^* + \vec{p}^* = \frac{M}{m_2}\vec{p}^*$

After collision:

$$\vec{q}_1 = \frac{m_1}{m_2}\vec{p}^* + \vec{q}^* \quad , \quad \vec{q}_2 = \vec{p}^* - \vec{q}^*$$

(use $\vec{p}_1 = m_1 \dot{\vec{R}} + \vec{p}^*$ and $\vec{p}_2 = m_2 \dot{\vec{R}} - p^*$ but put $p^* \to q^*$
and use $\dot{\vec{R}} = \frac{1}{m_2} p^*$)



Note 1) from $|\vec{p}^*| = |\vec{q}^*|$ vectors $\vec{p}^*$, $\vec{q}^*$ and $\vec{q}_2$ form an ~~[illegible]~~ isosceles triangle

Thus mag. and recoil angle of $\vec{q}_2$ given by cm quantities:

~~[illegible]~~ $\alpha = \frac{1}{2}(\pi - \theta^*)$

$$q_2 = 2 p^* \sin \tfrac{1}{2}\theta^* \quad \Rightarrow T_2 = \frac{q_2^2}{2m_2} = \frac{2p^{*2}}{m_2}\sin^2\frac{\theta^*}{2}$$

2) Lab KE. (KE of incoming particle) $\quad T = \frac{p_1^2}{2m_1} = \frac{M^2 p^{*2}}{2m_1 m_2^2}$ by $\vec{p}_1 = \frac{M}{m_2} p^*$

Fraction of total KE transferred:

$$\frac{T_2}{T} = \frac{4 m_1 m_2}{M^2} \sin^2 \frac{\theta^*}{2}$$

Max transfer: head on collision $(\theta^* = \pi)$ $\quad \frac{T_2}{T} = \frac{4m_1 m_2}{(m_1+m_2)^2}$

This can be close to unity only if $m_1 \simeq m_2$

e.g. proton - α-particle collision $\frac{m_1}{m_2} = 4$ or $\frac{1}{4}$

$$\frac{T_2}{T_{max}} = 64\%$$

electron proton $\frac{m_1}{m_2} = \frac{1}{1836}$ $\left(\frac{T_2}{T_{max}}\right) \approx 0.2\%$

lab scattering angle

com scattering angle

Also note $\tan \theta = \dfrac{\sin \theta^*}{m_1/m_2 + \cos \theta^*}$