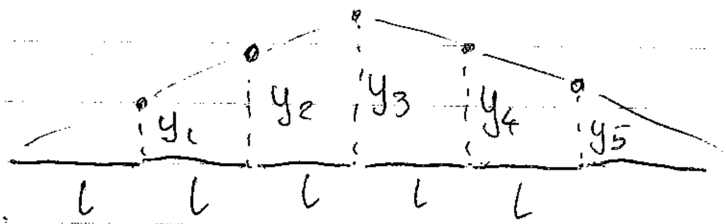
Oscillations of particles on a string (Sec 11.5) ①



transverse oscillations

- Light string of length $(n+1)l$
- n equal masses m
- Tension F
- generalised coords $y_1, y_2 \cdots y_n$

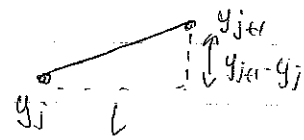
$$T = \tfrac{1}{2}m\left(\dot{y}_1^2 + \dot{y}_2^2 + \cdots \dot{y}_n^2\right)$$

$\Rightarrow$ generalized coords are orthogonal

Potential energy: ~~length~~ Consider string between j and $j+1$ masses.

Equilibrium length: l

When displaced: $l + \delta l = \sqrt{l^2 + (y_{j+1} - y_j)^2}$



$$\simeq l\left(1 + \tfrac{1}{2l^2}(y_{j+1} - y_j)^2\right)$$

(Also applies to end segments if $y_0 = y_{n+1} = 0$).

Work done against tension in increasing length $= F\delta l$

total $V = \frac{F}{2l}\left[y_1^2 + (y_2 - y_1)^2 + \cdots + (y_n - y_{n-1})^2 + y_n^2\right]$

Note: continuous string: $n \to \infty$, $l \to 0$

then $\frac{(y_{j+1} - y_j)^2}{l^2} \to y'^2$

Lagrange's equs

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}_j} = \frac{\partial L}{\partial y_j} \Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{y}_j} = -\frac{\partial V}{\partial y_j}$$

$$\ddot{y}_1 = -\frac{F}{2lm}\left(2y_1 + 2(-1)(y_2 - y_1)\right)$$

$$= -\frac{F}{lm}\left(+2y_1 - y_2\right)$$

$$\ddot{y}_2 = -\frac{F}{2ml}\left(2(y_2 - y_1) + 2(-1)(y_3 - y_2)\right)$$

$$= -\frac{F}{2ml}\left(4y_2 + 2y_1 - 2y_3\right)$$

$\vdots$

last one — not general n. $\longrightarrow$ $\ddot{y}_n = \frac{F}{ml}\left(y_{n-1} - 2y_n\right)$

Put $\omega_0^2 = \frac{F}{ml}$ and $y_j = A_j e^{i\omega t}$

$$\begin{bmatrix} 2\omega_0^2 & -\omega_0^2 & 0 & \cdots & \\ -\omega_0^2 & 2\omega_0^2 & -\omega_0^2 & \cdots & \\ 0 & -\omega_0^2 & 2\omega_0^2 & \cdots & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 2\omega_0^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix} = \omega^2 \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \vdots \\ A_n \end{bmatrix}$$

$n=1$ One normal mode $\omega^2 = 2\omega_0^2$

$n=2$ two " " Characteristic equation $(2\omega_0^2 - \omega^2)^2 - \omega_0^4 = 0$

$\omega^2 = \omega_0^2$ $A_1/A_2 = 1$

$\omega^2 = 3\omega_0^2$ $A_1/A_2 = -1$

③

$n=3$

Characteristic equ
$$\begin{vmatrix} 2\omega_0^2-\omega^2 & -\omega_0^2 & 0 \\ -\omega_0^2 & 2\omega_0^2-\omega^2 & -\omega_0^2 \\ 0 & -\omega_0^2 & 2\omega_0^2-\omega^2 \end{vmatrix} = 0$$

Cubic equ : $$(2\omega_0^2-\omega^2)^3 - 2\omega_0^4(2\omega_0^2-\omega^2)=0$$

roots : $2\omega_0^2$
$$\begin{cases} (2+\sqrt{2})\omega_0^2 \\ (2-\sqrt{2})\omega_0^2 \end{cases}$$

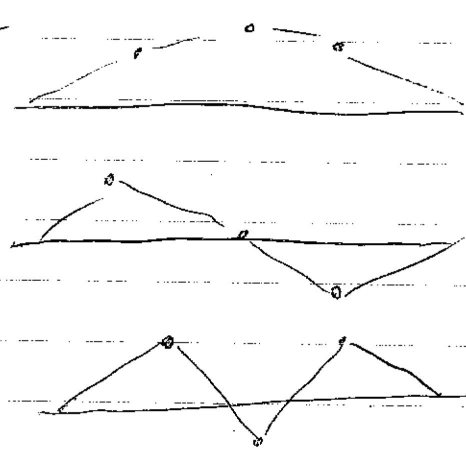
Normal modes

$\omega^2 = (2-\sqrt{2})\omega_0^2$ $\quad A_1 : A_2 : A_3 = 1 : \sqrt{2} : 1$

$\omega^2 = 2\omega_0^2$ $\quad A_1 : A_2 : A_3 = 1 : 0 : -1$

$\omega^2 = (2+\sqrt{2})\omega_0^2$ $\quad A_1 : A_2 : A_3 = 1 : -\sqrt{2} : 1$

~~Can also solve n=4~~



Can also solve $n=4$.

Continuum case Infinite no. of degrees of freedom

displacement $y(x,t)$ continuous function

Notation $\dot{y} = \frac{\partial y}{\partial t}$ $y' = \frac{\partial y}{\partial x}$

K.E. of small section of string dx : $\frac{1}{2}(\mu\, dx)\dot{y}^2$

$$T = \int_0^l \tfrac{1}{2}\mu \dot{y}^2\, dx$$

$$V = \int_0^l \tfrac{1}{2} F y'^2\, dx$$

$$L = \int_0^l \left(\tfrac{1}{2}\mu\dot{y}^2 - \tfrac{1}{2}F y'^2\right) dx$$

Useful to define $L = \int_0^l \mathcal{L}(y, \dot{y}, y')\, dx$

Lagrangian density.

Action : $I = \int_{t_0}^{t_1}\int_0^l \mathcal{L}(y, \dot{y}, y')\, dx\, dt$

small variation $\delta y(x,t)$, vanishes at t_0 and t_1 and at $x=0$ and $x=l$

$$\delta I = \int_{t_0}^{t_1}\int_0^l \left[\frac{\partial \mathcal{L}}{\partial y}\delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{\partial(\delta y)}{\partial t} + \frac{\partial \mathcal{L}}{\partial y'}\frac{\partial(\delta y)}{\partial x}\right] dx\, dt$$

Integrate by parts wrt t for 2nd term
wrt x for 3rd

$$\delta I = \int_{t_0}^{t_1}\int_0^l \left[\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right)\right]\delta y(x,t)\, dx\, dt$$

$$\frac{\partial \mathcal{L}}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{y}}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = 0$$

Stretched string:

$$\frac{\partial \mathcal{L}}{\partial y} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \dot{y}} = \mu \dot{y} \quad , \quad \frac{\partial \mathcal{L}}{\partial y'} = -Fy'$$

Lagrange's eqn:

$$\ddot{y} = c^2 y'' \qquad c^2 = \frac{F}{\mu}$$

partial differential eqn

Solution of wave eqn

$$y(x,t) = f(x - ct) \qquad \text{for any function } f \, !$$

i.e. let $x - ct = s$

$$\left\{ \begin{aligned} \dot{f} &= \frac{df}{ds}\frac{\partial s}{\partial t} = \frac{df}{ds}(-c) \\ \ddot{f} &= \frac{d^2f}{ds^2}\, c^2 \end{aligned} \right.$$

$$\left\{ \begin{aligned} y' &= \frac{df}{ds}\frac{\partial s}{\partial x} = \frac{df}{ds} \\ y'' &= \frac{d^2f}{ds^2} \end{aligned} \right.$$

so $c^2 y'' = \ddot{y}$

However need two arbitrary functions (2nd order eqn)

$$y = f(ct + x) + g(ct - x)$$

Normal modes?

$$y(x,t) = A(x)e^{i\omega t}$$

$$\Rightarrow \quad A''(x) + \underbrace{\frac{\omega^2}{c^2}}_{=k^2} A(x) = 0$$

So $\quad A_j \to A(x)$

Gen. sol.: $\quad A(x) = a\cos kx + b\sin kx$

If ends of string are fixed $\quad A(0) = A(l) = 0.$

$\Rightarrow \quad a = 0 \quad$ and $\quad \sin kl = 0$

$$\Rightarrow \quad k = \frac{n\pi}{l} \qquad n = 1, 2, 3, \ldots$$

Each n labels a different normal mode.

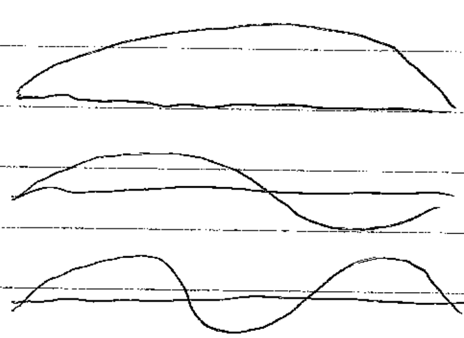
$$\omega = ck = \frac{n\pi c}{l}$$

Fundamental: $\quad \omega_1 = \frac{\pi c}{l} = \pi\sqrt{\frac{F}{Ml}}$

mass of whole string.

so n^{th} mode: $\quad y(x,t) = \mathrm{Re}\left(A_n e^{in\pi ct/l}\right)\sin\frac{n\pi x}{l}$ $\quad$ Standing wave

A_n = arbitrary complex constant



$n=1$

$n=2$

$n=3$

Gen. sol. is a superposition of normal modes. 7

Note that solution must be periodic with period $2l$:

If $y=0$ at $x=0$

$$f(ct) + g(ct) = 0 \quad \text{for } \underline{\text{all}}\ t.$$

$\Rightarrow$ f and g differ only in sign

$$\Rightarrow y = f(ct+x) - f(ct-x)$$

2nd condition $y=0$ at $x=l$.

$$f(ct+l) - f(ct-l) = 0 \qquad \text{for all } t$$

i.e. at a particular time, f at l and f at $-l$ are identical

$\Rightarrow$ f is periodic of period $2l$:

$$f(x+2l) = f(x).$$

Periodic so...
Fourier analyze

$$f(x) = \sum_{n=-\infty}^{+\infty} f_n e^{\frac{in\pi x}{l}}$$

sub into $y = f(ct+x) - f(ct-x)$

$$y(x,t) = \sum_{n=-\infty}^{+\infty} f_n \left(e^{in\pi(ct+x)/l} - e^{in\pi(ct-x)/l} \right)$$

$$= \sum_{n=-\infty}^{+\infty} 2i f_n e^{in\pi ct/l} \sin\frac{n\pi x}{l}$$

$f_n = f_{-n}^*$ for real soln. $\Rightarrow$ restrict sum to +ve n

$$y(x,t) = 2\,\mathrm{Re} \sum_{n=1}^{+\infty} A_n e^{in\pi ct/l} \sin\frac{n\pi x}{l} \qquad A_n = 2if_n$$

N.B. $n=0$ terms vanishes because $\sin(0)=0$.