Small oscillations and normal modes (1)

Chpt 11 in Kibble + Berkshire 5th Ed.

Near a position of equilibrium any system behaves like a SHO.
Let us generalize this to more than 1 degree of freedom.
We consider only conservative, ~~non~~holonomic systems described by n generalized coords $q_1, \dots q_n$

Can always choose pt. of equilibrium to be $q_1 = q_2 = \dots q_n = 0$

What does T look?

Further restrict attention to "natural systems" = all constraints are algebraic: t does not appear explicitly

For n=2 $\quad T = \frac{1}{2} a_{11} \dot{q}_1^2 + a_{12} \dot{q}_1 \dot{q}_2 + \frac{1}{2} a_{22} \dot{q}_2^2$

(Why? $\quad r_i = r_i(q_1, q_2, q_3, \dots, q_n ; \cancel{t})$

$$\frac{dr_i}{dt} = \sum_j \frac{\partial r_i}{\partial q_j} d\dot{q}_j + \cancel{\frac{\partial r_i}{\partial t}}$$

Then sub into $T = \sum \frac{1}{2} m \dot{r}^2 \dots$)

a_{ij} may be functions of q ... but for small q, can take them equal to constants (evaluated at equilibrium pt)

We can always choose "orthogonal" coordinates: $T = \sum_j a_{jj} \dot{q}_j^2$

e.g. $a_{12} = 0$ in n=2 case.

put $q_1' = q_1 + \frac{a_{12}}{a_{11}} q_2$.

$$\Rightarrow T = \frac{1}{2} a_{11} \left(\dot{q}_1' - \frac{a_{12}}{a_{11}} \dot{q}_2\right)^2 + a_{12} \left(\dot{q}_1' - \frac{a_{12}}{a_{11}} \dot{q}_2\right) \dot{q}_2 + \frac{1}{2} a_{22} \dot{q}_2^2$$

$$= \frac{1}{2} a_{11} \dot{q}_1'^2 - a_{11} \frac{a_{12}}{a_{11}} \dot{q}_1' \dot{q}_2 + \frac{1}{2} \frac{a_{12}^2}{a_{11}} \dot{q}_2^2 + a_{12} \dot{q}_1' \dot{q}_2 - \frac{a_{12}^2}{a_{11}} \dot{q}_2^2 + \frac{1}{2} a_{22} \dot{q}_2^2$$

$$T = \tfrac{1}{2} a_{11} \dot{q}_1'^2 + \tfrac{1}{2} a_{22}' \dot{q}_2^2$$

where $a_{22}' = a_{22} - \dfrac{a_{12}^2}{a_{11}}$

Can also put $\quad q_1'' = \sqrt{a_{11}}\, q_1' \quad , \quad \cancel{a_{22}}\, q_{22}'' = \sqrt{a_{22}'}\, q_2$

$$\Rightarrow \quad T = \tfrac{1}{2} \dot{q}_1''^2 + \tfrac{1}{2} \dot{q}_2''^2$$

General case (Gram-Schmidt orthogonalization procedure):

$$q_1' = q_1 + \frac{a_{12}}{a_{11}} q_2 + \dots + \frac{a_{1n}}{a_{11}} q_n.$$

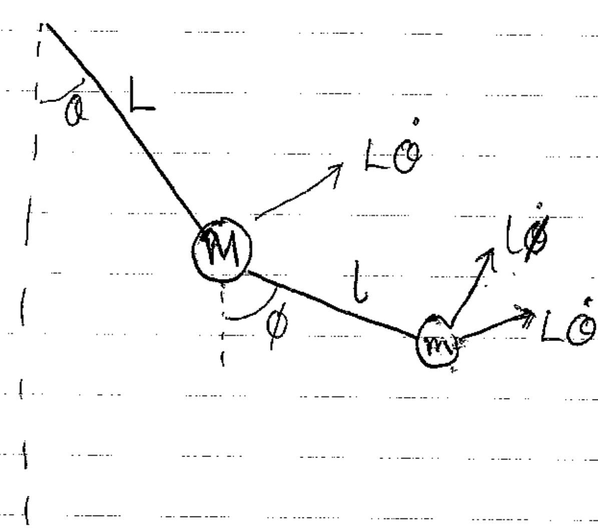
eliminate cross terms involving $\dot{q}_1$

etc. ....

$$\Rightarrow \quad \boxed{T = \sum_j \tfrac{1}{2} \dot{q}_j^2}$$

(drop primes)

Example Double pendulum



2 degrees of freedom

Velocity of $M = L\dot{\theta}$

" " $m = \vec{V}_1 + \vec{V}_2$

$$V_m = \sqrt{(\vec{V}_1+\vec{V}_2)\cdot(\vec{V}_1+\vec{V}_2)} = \sqrt{V_1^2 + V_2^2 + 2\vec{V}_1\cdot\vec{V}_2}$$

$$= \sqrt{L^2\dot{\theta}^2 + l^2\dot{\phi}^2 + 2Ll\dot{\theta}\dot{\phi}\cos(\phi-\theta)}$$

$$T = \tfrac{1}{2}ML^2\dot{\theta}^2 + \tfrac{1}{2}m\left[L^2\dot{\theta}^2 + l^2\dot{\phi}^2 + 2Ll\dot{\theta}\dot{\phi}\cos(\phi-\theta)\right]$$

For small angles $\cos(\phi-\theta) \simeq 1$

Nonorthogonal : $\dot{\theta}\dot{\phi}$ term

Orthogonal coords: displacements of two bobs :

$$x = L\theta \qquad\qquad y = L\theta + l\phi$$

$$\Rightarrow T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\dot{y}^2$$

2nd term in T:

$$\to 1$$

$$L^2\dot{\theta}^2 + l^2\dot{\phi}^2 + 2Ll\dot{\theta}\dot{\phi}\cos(\phi-\theta)$$

$$\Rightarrow \dot{x}^2 + (\dot{y}-\dot{x})^2 + 2\dot{x}(\dot{y}-\dot{x})$$

$$= \dot{x}^2 + \dot{y}^2 + \dot{x}^2 - 2\dot{x}\dot{y} + 2\dot{x}\dot{y} - 2\dot{x}^2$$

$$= \dot{y}^2$$

Finally $q_1 = \sqrt{M}\,x$, $q_2 = \sqrt{m}\,y$.

Now consider <u>potential energy</u>

Eqn of motion:

$$\ddot{q}_\alpha = \underbrace{-\frac{\partial V}{\partial q_\alpha}}_{=0}$$

zero at equilibrium

For $n=2$:

$$V = V_0 + (b_1 q_1 + b_2 q_2) + \left(\tfrac{1}{2} k_{11} q_1^2 + k_{12} q_1 q_2 + \tfrac{1}{2} k_{22} q_2^2\right) + \ldots$$

Equilibrium: $b_1 = b_2 = 0$ Set $V_0 = 0$ without changing eqn. of motion.

Then

$$\ddot{q}_1 = -k_{11} q_1 - k_{12} q_2$$
$$\ddot{q}_2 = -k_{21} q_1 - k_{22} q_2$$

(have put $k_{12} = k_{21}$)

<u>General case</u>:

$$V = \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} \tfrac{1}{2} k_{\alpha\beta} q_\alpha q_\beta$$

$k_{\alpha\beta} = k_{\beta\alpha}$ counts $\alpha \neq \beta$ twice...

so

$$\begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \vdots \\ \ddot{q}_n \end{bmatrix} = -\begin{bmatrix} k_{11} & k_{12} & \cdots & k_{1n} \\ k_{21} & k_{22} & \cdots & k_{2n} \\ \vdots & & & \\ k_{n1} & k_{n2} & \cdots & k_{nn} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_n \end{bmatrix}$$

Example double pendulum.

$$V = (M+m)gL(1-\cos\theta) + mgl(1-\cos\phi)$$

Small angle: $(1-\cos\theta) \to \frac{1}{2}\theta^2$

$$\Rightarrow V = \frac{1}{2}(M+m)gL\theta^2 + \frac{1}{2}mgl\phi^2 = \frac{(M+m)}{2L}g x^2 + \frac{mg}{2l}(y-x)^2$$

Eqns of motion

$$\begin{bmatrix} M\ddot{x} \\ m\ddot{y} \end{bmatrix} = \begin{bmatrix} -\frac{(M+m)g}{L} - \frac{mg}{l} & \frac{mg}{l} \\ \frac{mg}{l} & -\frac{mg}{l} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Need 4 arbitrary constants, e.g. $q_1, q_2, \dot{q}_1, \dot{q}_2$ at $t=0$

Normal modes

$$\boxed{q_\alpha = A_\alpha e^{i\omega t}}$$

All coords are oscillating with same freq. ω.

A_α = Complex constants.

i.e. simple harmonic motion.

$2n$ arbitrary constants

Physical solution is the real part

$$\Rightarrow -\omega^2 A_\alpha = -\sum_{\beta=1}^{n} k_{\alpha\beta} A_\beta$$

e.g. $n=2$

$$\omega^2 \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

eigenvalue equation

$$\Rightarrow \begin{bmatrix} k_{11}-\omega^2 & k_{12} \\ k_{21} & k_{22}-\omega^2 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Determinant vanishes for non-zero solutions

$$\begin{vmatrix} k_{11}-\omega^2 & k_{12} \\ k_{21} & k_{22}-\omega^2 \end{vmatrix} = (k_{11}-\omega^2)(k_{22}-\omega^2) - k_{12}^2 = 0$$

Characteristic eqn.

e.g. Double pendulum

$$\begin{bmatrix} \frac{(M+m)g}{ML} + \frac{mg}{Ml} & -\frac{mg}{Ml} \\ -\frac{g}{l} & \frac{g}{l} \end{bmatrix} \begin{bmatrix} A_x \\ A_y \end{bmatrix} = \omega^2 \begin{bmatrix} A_x \\ A_y \end{bmatrix}$$

Characteristic eqn:

$$\omega^4 - \frac{M+m}{M}\left(\frac{g}{L}+\frac{g}{l}\right)\omega^2 + \frac{M+m}{M}\frac{g^2}{Ll} = 0$$

obtain ratio by subbing ω back into eigenvalue equation

Limiting cases

$M >> m$

upper pendulum is nearly stationary

$$\omega^2 \approx \frac{g}{l} \qquad \frac{A_x}{A_y} \approx \frac{m}{M}\frac{L}{l-L}$$

$$\omega^2 \approx \frac{g}{L} \qquad \frac{A_x}{A_y} \approx \frac{L-l}{L}$$

Comparable amplitudes

(assume $l \neq L$)

$m >> M$

$$\omega^2 = \frac{g}{L+l}$$

swings like a single pendulum of length $L+l$

$$\frac{A_x}{A_y} \approx \frac{L}{L+l}$$

$$\omega^2 = \frac{m}{M}\left(\frac{g}{L}+\frac{g}{l}\right)$$

lower bob almost stationary, upper one executes fast oscillation

$$\frac{A_x}{A_y} \approx -\frac{m}{M}\frac{L+l}{L}$$

Superposition principle

Eqns of motion are linear $\Rightarrow$ any superposition of solution is also a solution.

General solution is superposition of normal modes.
Note: normal modes are orthogonal.
Example: If ω^2 and ω'^2 are roots for $n=2$ then gen sol. is real part of

$$q_1 = A_1 e^{i\omega t} + A_1' e^{i\omega' t}$$
$$q_2 = A_2 e^{i\omega t} + A_2' e^{i\omega' t}$$

Ratios $\frac{A_1}{A_2}$ and $\frac{A_1'}{A_2'}$ are fixed by plugging each eigenvalue (ω^2 and ω'^2) into eigenvalue matrix eqn.

Note: all k_{ij} are real $\Rightarrow$ $\frac{A_1}{A_2}$ is real

$\Rightarrow$ A_1 and A_2 have same phase or are π out of phase

$\Rightarrow$ q_1 and q_2 not only oscillate with same frequency, but also in or directly out of phase.

A_1 and A_2 can have a common arbitrary complex factor, which fixes overall amplitude and phase of normal mode (fit to boundary conditions) $\Rightarrow$ 2 arbitrary real constants for each mode.

$\Rightarrow$ 4 constants to set in above solution

Same for $n = 3$

$$\begin{vmatrix} k_{11} - \omega^2 & k_{12} & k_{13} \\ k_{21} & k_{22} - \omega^2 & k_{23} \\ k_{31} & k_{32} & k_{33} - \omega^2 \end{vmatrix} = 0$$

- all 3 roots are real
- All ratios of amplitudes are real (so all coords oscillate exactly in or exactly out of phase)
- Each normal mode has just two arbitrary real constants

<u>Stability</u> $\begin{vmatrix} k_{11} - \omega^2 & k_{12} \\ k_{12} & k_{22} - \omega^2 \end{vmatrix} = 0$

$n=2$ case, discriminant : $(k_{11} - k_{22})^2 + 4 k_{12}^2 > 0$

i.e. $\sqrt{b^2 - 4ac}$

$\Rightarrow$ two real roots ω^2 and ω'^2

<u>Condition for stability</u>: both roots +ve

If one root is -ve : $\omega^2 = -\gamma^2$, say.

$$\Rightarrow \quad q_\alpha = A_\alpha e^{\gamma t} + B_\alpha e^{-\gamma t}$$

$\Rightarrow$ exponential increase with time.