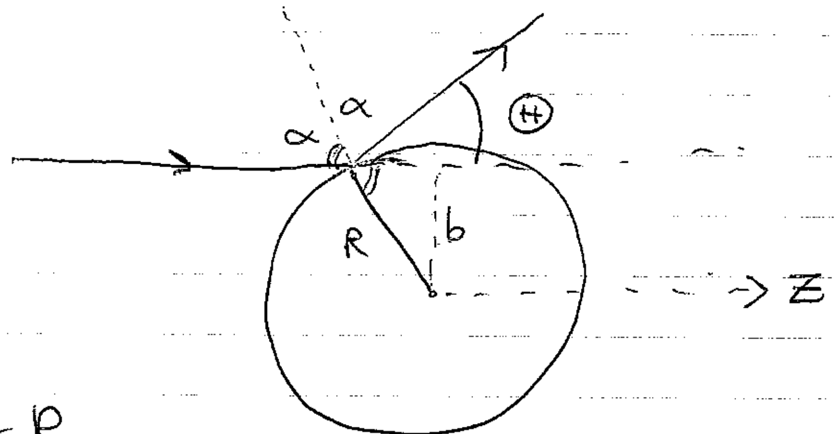
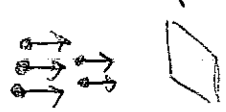
Scattering cross-section

We want to know the angular distribution of scattering particles



Hard sphere of radius R
→ force is central and elastic

Particle flux $= f =$ # of particles crossing unit area in unit time



# of particles which strike target in unit time $= W = f\sigma$

$$\sigma = \pi R^2$$

Consider one particle, with velocity v, and impact parameter b.
Then, it hits target at angle α to the normal.

$$b = R\sin\alpha$$

Elastic collision and central force:

$E' = E \Rightarrow v = v'$

$J' = J$ ($r \times p$ is same) $\Rightarrow$ angle on = angle off

Axial symmetry: particle moves in plane ϕ is const

Proof of "angle on = angle off"
is reversed

part of $\vec p$ perp to radius $\vec R$ is constant. Other component is reversed
cons. of A.M.

Deflection angle $\Theta = \pi - 2\alpha$

$$b = R\cos\frac{\Theta}{2}$$ i.e. 

$\frac{\pi}{2} - \alpha = \frac{\Theta}{2}$

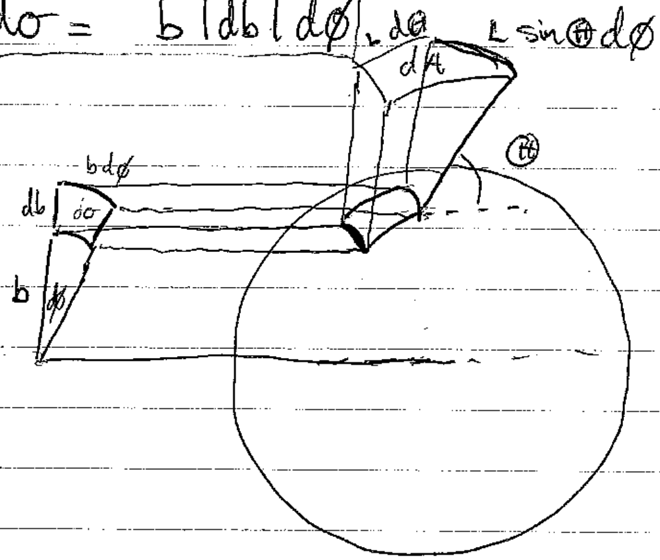
How many particles scattered in direction specified by (Θ, ϕ) in range $d\Theta$, $d\phi$ about this direction?

Particles scattered through angles between Θ and $\Theta + d\Theta$ arrived with impact parameter between b and $b + db$

$$db = -\tfrac{1}{2} R \sin\frac{\Theta}{2}\, d\Theta$$

Consider a cross-section of the incoming beam. Our particles are those which cross small element of area

$$\boxed{d\sigma = b\,|db|\,d\phi}$$



Insert b and db: $d\sigma = R\cos\frac{\Theta}{2}\,\left|-\tfrac{1}{2} R \sin\frac{\Theta}{2}\, d\Theta\right|\, d\phi$ but $\sin a \cos a = \tfrac{1}{2}\sin 2a$

$$= \tfrac{1}{4} R^2 \sin\Theta\, d\Theta\, d\phi$$

Rate at which particles cross this area and therefore emerge in given angular range is $dw = f\, d\sigma$

Put detector at large distance $L \gg R$ from target

Express rate in ~~the~~ terms of cross-sectional area dA of detector

$$dA = L\, d\Theta \times L \sin\Theta \, d\phi$$ (element of area on sphere of radius L)

Define solid angle subtended at origin by the area dA to be

$$d\Omega \equiv \sin\Theta \, d\Theta \, d\phi$$

so $$dA = L^2 \underbrace{d\Omega}_{\text{steradians}}$$

N.B. $$\iint d\Omega = \frac{1}{L^2}\iint dA = \int_0^{\pi} \sin\Theta \, d\Theta \int_0^{2\pi} d\phi = 4\pi$$

N.B. $\int_0^{\pi} \sin\Theta \, d\Theta = -\cos\Theta \Big|_0^{\pi} = -(-1-1) = 2$

Important quantity is $\dfrac{d\sigma}{d\Omega}$ "differential cross-section"

Rate at which particles enter detector is (by $dw = f\, d\sigma$ and $dA = L^2 d\Omega$)

$$dw = \underbrace{f}_{\text{incident flux}} \; \underbrace{\frac{d\sigma}{d\Omega}}_{\text{intrinsic scattering probability, independent of solid angle subtended by detector or flux}} \; \underbrace{\frac{dA}{L^2}}_{\text{solid angle detector}}$$

Note $\dfrac{d\sigma}{d\Omega}$ has dimensions of area (sq metres per steradian)

Alternatively, DEFINE $\dfrac{d\sigma}{d\Omega} = \dfrac{dw}{f\, d\Omega}$ = ratio of # of scattered particle per unit solid angle to number of incoming particles per unit area

here $$\frac{d\sigma}{d\Omega} = \frac{R^2}{4}$$ by $d\sigma = \frac{1}{4}R^2 \sin\Theta\, d\Theta\, d\phi$, $d\Omega = \sin\Theta \, d\Theta \, d\phi$

Isotropic ! N.B. $\sigma = \pi R^2 = \int \frac{d\sigma}{d\Omega} d\Omega$

$\rightarrow$ alpha particles ($Z=2, A=4$) from gold ($Z=79, A=276$)

Rutherford scattering

particle of charge q and mass m scattered by a fixed point charge q'

Impact parameter: in terms of scattering angle $\underbrace{b = a \cot \frac{\Theta}{2}}$ where $a = \frac{q q'}{4\pi\epsilon_0 m v^2}$

go back to page p8 of orbits notes!

$$\Rightarrow \quad db = -\frac{a\, d\Theta}{2\sin^2 \frac{\Theta}{2}}$$

then $$d\sigma = \frac{a^2 \cos\frac{1}{2}\Theta\, d\Theta\, d\phi}{2\sin^3\frac{\Theta}{2}}$$
$(= b\,|db|\,d\phi$

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{4\sin^4\frac{\Theta}{2}}$$

Rutherford scattering cross-section

N.B. Quantum mechanics gives same result, luckily for Rutherford!

i.e. multiply ~~by~~ $\frac{\sin\frac{\Theta}{2}}{\sin\frac{\Theta}{2}}$, but $\cos\frac{\Theta}{2}\sin\frac{\Theta}{2} = \frac{1}{2}\sin\Theta$

<u>Note</u> 1. Unlike hard sphere, $\frac{d\sigma}{d\Omega}$ depends on velocity and scattering angle

2. increases rapidly with increasing charge

3. for α-particle scattering from a nucleus of atomic number Z, (# of protons in nucleus)

$q q' = 2Ze^2 \Rightarrow$ # of scattered particles goes as Z^2

(2 protons in α-particle + 2 neutron)

4. To investigate structure of atom at small distances, we must use high velocities (small a) and look at large-angle scattering ($\Theta \gtrsim \pi/2$), corresponding to particles with small impact parameter. However, Rutherford s.c. is small for large angles? Nevertheless Rutherford S.C. is finite for large Θ and this is an indication of very strong forces at short distances. If the nuclear charge were spread over large volume force would decrease once inside atom. $\Rightarrow$ no scattering at large angles. NB $r_{nucleus} \sim 8\times10^{-15}$ m i.e. factor of 10^4 smaller than atom itself

5. Electrons are very light, ie $\frac{1}{1836}$ ~~proton~~ mass, so they don't deflect α-particle much - so scattering just gives info about nucleus.

Rutherford's conclusion: $\frac{1}{r^2}$ force law valid down to very small distances $\Rightarrow$ +ve charge concentrated in very small volume rather than whole atom.
i.e. he found agreement with point charge result except for the very largest angles corresponding to $b \sim 10^{-14}$ m

Strange thing

$$\int \frac{d\sigma}{d\Omega}\, d\Omega = \infty \, !$$

This is due to infinite range of $\frac{1}{r}$ potential.
No matter how far away, particles are still scattered through a small but non-zero angle $\Rightarrow$ # of particles scattered through any angle is indeed infinite

Can calculate # of particles through angle greater than some small lower limit Θ_0. These have $b < b_0 = a \cot \frac{1}{2}\Theta_0$

then $$\sigma(\Theta > \Theta_0) = \pi b_0^2 = \pi a^2 \cot^2 \frac{\Theta_0}{2}$$

(Can also verify by integrating $\frac{d\sigma}{d\Omega}$).