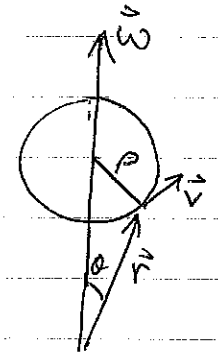
Rotating frames (Chapter 5 in Kibble)

For some problems it is useful to use non-inertial frames, e.g. frames attached to Earth's surface.

Rate of change of a vector: $\frac{d\vec{a}}{dt} = \vec{\omega} \wedge \vec{a}$

Define angular velocity vector $\vec{\omega}$ to be $\vec{\omega} = \omega \hat{n}$

defined by right hand rule (direction a right-handed screw would move when turned in direction of rotation)



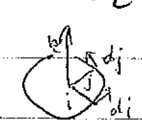
like angular momentum, $\vec{\omega}$ is an axial vector.

$$\vec{v} = \vec{\omega} \wedge \vec{r} \qquad \text{ie. } |\vec{v}| = \omega\rho = \omega r \sin\alpha$$

$\vec{r}$ ~~is not necessarily~~ could be the position vector of a point on a rotating body, but can also be any vector fixed in rotating body.

ie. $$\boxed{\frac{d\vec{a}}{dt} = \vec{\omega} \wedge \vec{a}}$$

Consider the vector $\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}$ specified wrt rotating axes $\hat{i}, \hat{j}, \hat{k}$.



Two kinds of rate of change:

$\frac{d\vec{a}}{dt}$ = rate of change of $\vec{a}$ as measured by inertial observer at rest relative to origin

$\dot{\vec{a}}$ = rate of change " " " " an observer rotating with solid body i.e. temporarily suspend convention $\frac{d\vec a}{dt}$ and $\dot{\vec a}$ mean same thing

What is relation between $\dot{\vec a}$ and $\frac{d\vec a}{dt}$?

The Two observers disagree about rates of change of vectors, but agree on rates of change of scalars, including components a_x, a_y, a_z.

~~For an inertial observer~~

$$\Rightarrow \quad \frac{da_x}{dt} = \dot a_x \quad , \quad \frac{da_y}{dt} = \dot a_y \quad , \quad \frac{da_z}{dt} = \dot a_z$$

Observer on rotating body : $\dot{\vec a} = \dot a_x \hat\imath + \dot a_y \hat\jmath + \dot a_z \hat k$

inertial observer :
$$\frac{d\vec a}{dt} = \left(\frac{da_x}{dt}\hat\imath + \frac{da_y}{dt}\hat\jmath + \frac{da_z}{dt}\hat k\right) + \left(a_x \frac{d\hat\imath}{dt} + a_y \frac{d\hat\jmath}{dt} + a_z \frac{d\hat k}{dt}\right)$$
$$= \left(\dot a_x \hat\imath + \dot a_y \hat\jmath + \dot a_z \hat k\right) + \omega \wedge \left(a_x \hat\imath + a_y \hat\jmath + a_z \hat k\right)$$

2nd bracket: because $\frac{d\hat\imath}{dt} = \vec\omega \wedge \hat\imath$ etc

So $$\boxed{\frac{d\vec a}{dt} = \dot{\vec a} + \vec\omega \wedge \vec a}$$

change within rotating frame

change due to rotating frame.

Can reverse argument: if $\vec a$ satisfies $\frac{d\vec a}{dt} = \vec\omega \wedge \vec a$ then it must be a vector of constant length rotating with ang. velocity $|\vec\omega|$ about direction of $\vec\omega$.

<u>Proof</u>: introduce frame rotating with $\vec\omega$. But by above formula $\dot{\vec a} = 0$, so $\vec a$ is fixed in rotating frame.

Centripetal and Coriolis accelerations (fictitious forces)

How about $\dfrac{d^2\vec{r}}{dt^2}$ vs $\ddot{r}$?

Velocity in inertial frame : $\vec{V} = \dfrac{d\vec{r}}{dt} = \dot{\vec{r}} + \vec{\omega} \wedge \vec{r}$ (A)

Also Apply to get rate of change of $\vec{V}$

$$\frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2} = \dot{\vec{V}} + \vec{\omega} \wedge \vec{V} \qquad (B)$$

but from (A) we have

$$\dot{\vec{V}} = \ddot{\vec{r}} + \vec{\omega} \wedge \dot{\vec{r}} \qquad \text{(assuming } \vec{\omega} \text{ is const)}$$

And using (A) again $\vec{\omega} \wedge \vec{V} = \omega \wedge \dot{\vec{r}} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{r})$

sub into (B)

$$\frac{d^2\vec{r}}{dt^2} = \ddot{\vec{r}} + \underbrace{2\vec{\omega} \wedge \dot{\vec{r}}}_{\substack{\text{Coriolis} \\ \text{acceleration}}} + \underbrace{\vec{\omega} \wedge (\vec{\omega} \wedge \vec{r})}_{\substack{\text{Centripetal} \\ \text{acceleration}}}$$

Centripetal force is directed inwards : towards axis of rotation and perpendicular to it.

Proof $\quad \omega \wedge (\omega \wedge r) = (\vec{\omega} \cdot \vec{r})\,\vec{\omega} - \omega^2 \vec{r} \qquad a \wedge (b \wedge c) = \vec{b}\, a \cdot c - \vec{c}\, a b$

Take dot product with $\vec{\omega}$: $\quad (\vec{\omega} \cdot \vec{r})(\vec{\omega} \cdot \vec{\omega}) - \omega^2 \vec{r} \cdot \vec{\omega} \quad = 0$

Can add additional forces eg. gravity + others

$$m\frac{d^2\vec{r}}{dt^2} = m\vec{g} + \vec{F}$$

$$\Rightarrow \quad m\ddot{\vec{r}} = m\vec{g} + \vec{F} \quad \underbrace{\underbrace{-2m\,\vec{\omega}\wedge\dot{\vec{r}}}_{\text{Coriolis}} \; \underbrace{-\,m\,\vec{\omega}\wedge(\vec{\omega}\wedge\vec{r})}_{\text{centrifugal}}}_{\text{fictitious forces}}$$

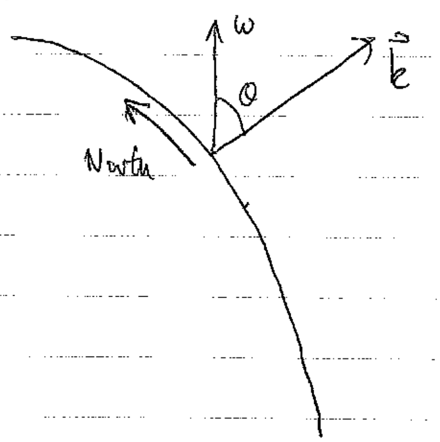
Foucault's pendulum (example of Coriolis force)

Ordinary pendulum, free to swing in any direction: must be symmetric, periods of oscillation in all directions are equal.

If amplitude is small : ~~[illegible]~~ two-dimensional SHM.
Can ignore vertical component of $\vec{F}_{\text{Coriolis}}$ - merely provides a correction to $\vec{g}$ whose sign alternates every half period - even when $\dot{x} = 10\ \mathrm{m\,s^{-1}}$, $2\omega\dot{x} = 1.5\ \mathrm{mm\,s^{-2}} \ll g$.
~~Can take~~ For small amplitude $\dot{z} \approx 0$ - motion of Bob is almost horizontal.

Coord system fixed on Earth

θ = colatitude (angle measured from north pole)



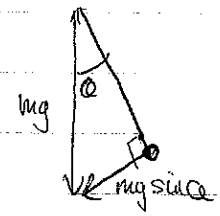
$\hat{\imath}$ is east

$\hat{\jmath}$ is north

$\hat{k}$ is up (local normal to surface)

$$\vec{\omega} = (0,\ \omega\sin\theta,\ \omega\cos\theta)$$

Then, equs of motion



$$F = ma$$

$$mg\sin\theta = ml\ddot{\theta}$$

$\Rightarrow$

$$\ddot{x} = -\frac{g}{l}x + 2\omega\dot{y}\cos\theta$$

$$\ddot{y} = -\frac{g}{l}y - 2\omega\dot{x}\cos\theta$$

i.e. $\vec{F}_c = -2m\,\vec{\omega}\wedge\dot{\vec{r}}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \omega\sin\theta & \omega\cos\theta \\ \dot{x} & \dot{y} & \underbrace{\dot{z}}_{=0} \end{vmatrix}$$

$$= \hat{i}(-\dot{y}\,\omega\cos\theta) - \hat{j}(-\dot{x}\,\omega\cos\theta)$$

Can write above as $\ddot{\vec{r}} = -\frac{g}{l}\vec{r} - 2\omega\cos\theta\,\hat{k}\wedge\dot{\vec{r}}$

If pendulum is at north pole ($\theta = 0$): pendulum swings in ~~in a non-rotating frame~~ fixed direction, Earth rotates beneath it. Relative to earth, oscillation plane rotates around vertical with angular ~~velocity~~ speed $-\omega$.

At any other latitude: $\vec{\omega} \rightarrow \omega\cos\theta\,\hat{k}$ i.e. vertical component of $\vec{\omega}$.

$\Rightarrow$ plane of oscillation of pendulum rotates with ang. ~~velocity~~ speed $-\Omega = -\omega\cos\theta$ about vertical.

In fact it is Earth's surface that rotates about vertical at rate $\omega \cos\theta$ (and about north-south axis at rate $\omega \sin\theta$, but this leads to vertical component of Coriolis force which we ignore).

Explicit solution

~~Define~~ Let $z = x + iy$

↑ not vertical coord!

then $\dot{y} - i\dot{x} = -i\dot{z}$

$i \times$ y-eqn + x-eqn : $\ddot{z} + 2i\Omega\dot{z} + \omega_0^2 z = 0$

$\Omega = \omega\cos\theta \qquad \omega_0^2 = g/l$

Soln: $z = Ae^{pt}$

$\Rightarrow \quad p^2 + 2i\Omega p + \omega_0^2 = 0$

$\Rightarrow \quad p = -i\Omega \pm i\omega_1 \qquad \omega_1 = \sqrt{\omega_0^2 + \Omega^2}$

$\Rightarrow \quad z = A e^{-i(\Omega - \omega_1)t} + B e^{-i(\Omega + \omega_1)t}$

if pendulum is released from rest at position $(a,0)$ at $t=0$ then ~~[illegible]~~ $A = B = \frac{a}{2}$

$\Rightarrow \quad z = a e^{-i\Omega t} \cos\omega_1 t$

or $\quad x = a\cos\Omega t \cos\omega_1 t$

$\quad y = -a\sin\Omega t \cos\omega_1 t$

but $\Omega << \omega_0$, $\omega_0 \simeq \omega_1$

So bob oscillates between extreme pts
$\pm a(\cos \Omega t, \sin \Omega t)$

i.e. oscillation with amplitude a in plane rotating at angular velocity $-\Omega$.

Period $= \dfrac{2\pi}{\omega \cos\theta}$ (34 hrs at 45°)