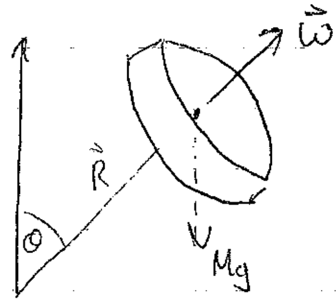
Alternative view to Spring's notes
— can skip this

Precession of a symmetric top

initially $\hat{e}_1, \hat{e}_2, \hat{e}_3 = \hat{i}, \hat{j}, \hat{k}$



generalized coords!

Euler's angles (θ, ϕ, ψ)

1) rotate by ϕ about $\hat{k}$
$\Rightarrow \hat{e}_1'', \hat{e}_2', \hat{k}$

2) rotate by θ about $\hat{e}_2'$
$\Rightarrow \hat{e}_1', \hat{e}_2', \hat{e}_3$

3) rotate by ψ about $\hat{e}_3$
$\Rightarrow \hat{e}_1, \hat{e}_2, \hat{e}_3$

General kinetic energy:

$$T = \tfrac{1}{2} I_1 \dot{\phi}^2 \sin^2\theta + \tfrac{1}{2} I_1 \dot{\theta}^2 + \tfrac{1}{2} I_3 (\dot{\psi} + \dot{\phi}\cos\theta)^2$$

$$\left(= \tfrac{1}{2} I_1 \omega_1^2 + \tfrac{1}{2} I_2 \omega_2^2 + \tfrac{1}{2} I_3 \omega_3^2\right)$$

$$V = MgR\cos\theta$$

$$L = \tfrac{1}{2} I_1 \dot{\phi}^2 \sin^2\theta + \tfrac{1}{2} I_1 \dot{\theta}^2 + \tfrac{1}{2} I_3 (\dot{\psi} + \dot{\phi}\cos\theta)^2 - MgR\cos\theta$$

Lagrange equ for θ:

$$\frac{d}{dt}(I_1 \dot{\theta}) = I_1 \dot{\phi}\sin\theta\cos\theta - I_3 (\dot{\psi} + \dot{\phi}\cos\theta)\dot{\phi}\sin\theta + MgR\sin\theta$$

Does not involve ϕ, or ψ $\Rightarrow$ p_ψ and p_ϕ are constants of motion

$$\frac{d}{dt}\left[I_1 \dot{\phi}\sin^2\theta + I_3 (\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta\right] = 0$$

$$\frac{d}{dt}\left[I_3 (\dot{\psi} + \dot{\phi}\cos\theta)\right] = 0$$

$\hookrightarrow$ so $\omega_3 = \dot{\psi} + \dot{\phi}\cos\theta = \text{const.}$

component of ang. velocity about symmetry axis.

Consider first steady precession at const. angle θ.

If θ is constant then $\dot{\phi}$ and $\dot{\psi}$ must also be constant

$\Rightarrow$ top precesses around vertical with const. ang. velocity $\dot{\phi} = \Omega$.

To find relation between Ω and ω_3 use eqn for $\frac{d}{dt}(I_1 \dot{\theta}) = 0$

$$I_1 \Omega^2 \cos\theta - I_3 \omega_3 \Omega + MgR = 0$$

quadratic eqn : there is a minimum value of ω_3 for which ~~the~~ roots are real

$$a x^2 + b x + c = 0$$

$$a = I_1 \cos\theta \qquad b = -I_3 \omega_3 \qquad c = MgR$$

$$b^2 - 4ac = I_3^3 \omega_3^2 - 4 I_1 \cos\theta \, MgR > 0$$

So $\quad I_3^2 (\omega_3^{min})^2 = 4 I_1 MgR \cos\theta$

If it spins more slowly it begins to wobble

If spins faster then 2 values of Ω.

Special case $\quad \omega_3$ large

$$\Omega \simeq \begin{cases} \dfrac{MgR}{I_3 \omega_3} & < \omega_3 \\[2ex] \dfrac{I_3 \omega_3}{I_1 \cos\theta} & \sim \omega_3 \end{cases}$$

result we obtained before

rapid precession where gravity is negligible
ie. precession of a free body (no gravity)

If $\Theta > \frac{\pi}{2}$ then steady precession is possible for any ω_3

If $\omega_3 = 0$ we find angular velocities of a compound pendulum swinging in a circle

$$\Omega = \pm \sqrt{\frac{MgR}{I_1 |\cos\Theta|}}$$

<u>General motion of symmetric top</u> (gyroscope)

Generalized momenta $\quad p = \frac{\partial L}{\partial \dot{q}}$

$$P_\phi = I_1 \dot{\phi} \sin^2\Theta + I_3 (\dot{\psi} + \dot{\phi}\cos\Theta)\cos\Theta \quad = \text{const}$$

$$P_\Theta = I_1 \dot{\Theta}$$

$$P_\psi = I_3 (\dot{\psi} + \dot{\phi}\cos\Theta) \quad = \text{Const.}$$

Solve for $\dot{\phi}, \dot{\Theta}, \dot{\psi}$

$$\dot{\phi} = \frac{P_\phi - P_\psi \cos\Theta}{I_1 \sin^2\Theta}$$

$$\dot{\Theta} = P_\Theta / I_1$$

$$\dot{\psi} = \frac{P_\psi}{I_3} - \frac{(P_\phi - P_\psi\cos\Theta)}{I_1 \sin^2\Theta}\cos\Theta$$

Construct Hamiltonian:

$$H = \sum_\alpha p_\alpha \dot{q}_\alpha - L$$, or, easier $H = T + U$ and use above equations in

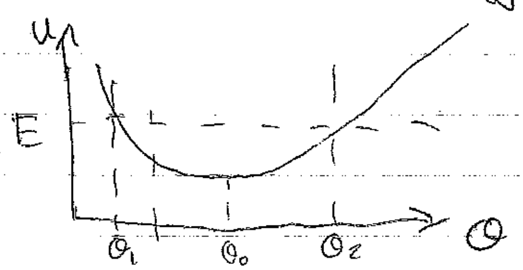
$$H = \frac{(P_\phi - P_\psi \cos\theta)^2}{2I_1 \sin^2\theta} + \frac{P_\theta^2}{2I_1} + \frac{P_\psi^2}{2I_3} + MgR\cos\theta$$

Check: $\dot{q}_\alpha = \frac{\partial H}{\partial P_\alpha}$? ✓ should reproduce above equations for $\dot\phi, \dot\theta, \dot\psi$

2 consts of motion $\Rightarrow$ only one degree of freedom θ.

$$H = \frac{P_\theta^2}{2I_1} + U(\theta)$$

effective potential energy: $$U(\theta) = \frac{(P_\phi - P_\psi\cos\theta)^2}{2I_1\sin^2\theta} + \frac{P_\psi^2}{2I_3} + MgR\cos\theta$$



Hamilton's equations for θ and P_θ:

$$\dot\theta = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{I_1}$$ ~~but $P_\theta = I_1\dot\theta$~~

$$\dot{P}_\theta = -\frac{\partial H}{\partial \theta} = -\frac{dU}{d\theta}$$ but $P_\theta = I_1\dot\theta$

so $$I_1\ddot\theta = -\frac{dU}{\partial\theta}$$ .... Complicated to solve

Qualitative features given by energy conservation

$$\frac{p_\theta^2}{2I_1} + U(\theta) = E$$

Angles θ at which $\dot{\theta} = 0$ satisfy $U(\theta) = E$ (turning pts)

Examine $U(\theta)$

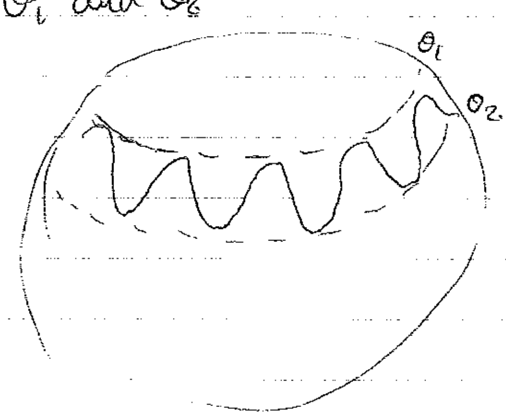
$U(\theta) \to +\infty$ when $\theta = 0$ or π

Can be shown there is only one minimum (equilibrium pt where θ = const $\Rightarrow$ steady precession.

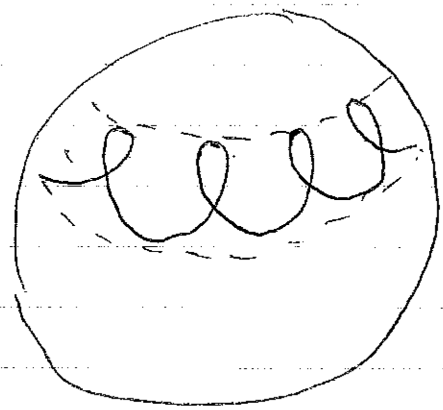
Otherwise θ oscillates ~~between~~ around minimum i.e. between θ_1 and θ_2.

From eqns for $\dot{\theta}, \dot{\phi}, \dot{\psi}$: $\dot{\phi} = 0$ when $\cos\theta = \frac{p_\phi}{p_\psi}$

1) If this angle lies outside range θ_1, θ_2 (or if $|p_\phi/p_\psi| > 1$) then $\dot{\phi}$ never vanishes and axis precesses round the vertical in a fixed direction, and wobbles up and down between θ_1 and θ_2

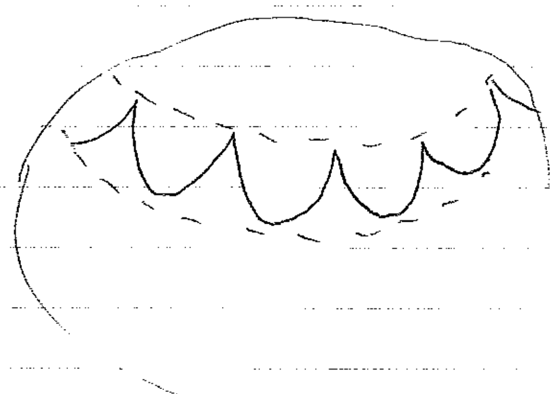


2) If $\theta_1 < \arccos\left(\frac{p_\phi}{p_\psi}\right) < \theta_2$
axis moves in loops



angular velocity has different sign ~~at~~ near top and bottom of loops

3) limiting case $\arccos\left(\frac{p_\phi}{p_\psi}\right) = \theta_1$
Loops shrink to cusps. Axis comes instantaneously to rest at cusp



Get this motion if top is set spinning with its axis initially at rest.

Impossible to have cusps at bottom: they correspond to minima of T, and motion must always be below such points.
A top set spinning with its axis stationary cannot rise without increasing its energy.