Rigid Body dynamics

Size and shape don't change.

$$\vec P = \sum m \dot{\vec r} \qquad \vec J = \sum m \vec r \wedge \dot{\vec r}$$

omit indices

Position of COM

Motion of COM: $\boxed{\dot{\vec P} = M\ddot{\vec R} = \sum \vec F}$ ← external forces

Rotational Motion: $\boxed{\dot{\vec J} = \sum \vec r \wedge \vec F}$ ← External forces

Assume internal forces are central.

i.e. $$\dot{\vec J} = \underbrace{\sum_i m_i \vec r_i \wedge \ddot{\vec r}_i}_{} = \underbrace{\sum_i \sum_j \vec r_i \wedge \vec F_{ij}}_{=0} + \sum_i \vec r_i \wedge \vec F_i$$

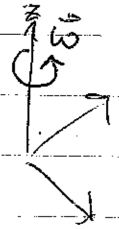
N.B. $\dot{\vec r} \wedge \dot{\vec r} = 0$ so only $\vec r \wedge \ddot{\vec r}$ remains

$= 0$ if forces are central

i.e. $$\vec r_1 \wedge \vec F_{12} + \vec r_2 \wedge \vec F_{21} = \vec r_1 \wedge \vec F - \vec r_2 \wedge \vec F = \vec r \wedge \vec F \;\hat{=}\; 0$$

$F_{21} = -F_{12}$ if central

Rotation about an axis



cylindrical polars (ρ, ϕ, z) $\quad \omega = \dot\phi$

$$J_z = \sum m \rho v_\phi = \sum m \rho^2 \omega = I\omega$$

$v_\phi = r\omega$

$$I = \sum m\rho^2 \quad \text{moment of inertia}$$

$$\dot J_z = I\dot\omega = \sum \rho F_\phi$$

I constant   <u>eqn of motion of rotating body</u>

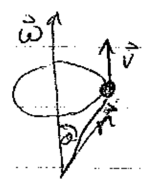
$$T = \sum \tfrac12 m (\rho\dot\phi)^2 = \tfrac12 I \omega^2$$

Reaction at the axis [Not covered in class]

$\dot{J}_z = I\dot{\omega} = \sum \rho F_\phi$ does not include reaction of axis because the axis is at $\rho = 0$ and cannot exert a torque.

Go back to $\dot{\vec{P}} = M\ddot{\vec{R}} = \underbrace{\vec{Q}}_{\text{force on body at axis}} + \underbrace{\sum \vec{F}}_{\text{all other forces}}$

Now, $\vec{v} = \vec{\omega} \wedge \vec{r} \Rightarrow \dot{\vec{R}} = \vec{\omega} \wedge \vec{R}$

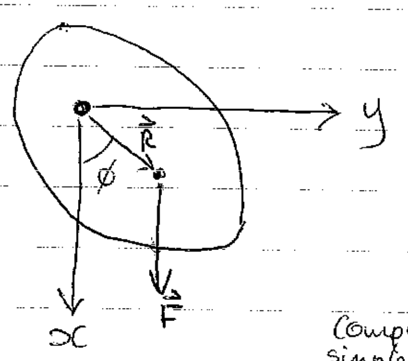


$$\Rightarrow \ddot{\vec{R}} = \dot{\vec{\omega}} \wedge \vec{R} + \vec{\omega} \wedge \dot{\vec{R}} = \underbrace{\dot{\vec{\omega}} \wedge \vec{R}}_{\substack{\text{tangential acceleration} \\ \text{in } \phi \text{ direction}}} + \underbrace{\vec{\omega} \wedge (\vec{\omega} \wedge \vec{R})}_{\text{radial acceleration}}$$

Combine above eqns to find $\vec{Q}$

Example Solid body pendulum

Take z-axis as axis of rotation



$$\vec{F} = \overset{x}{(Mg}, \overset{y}{0}, \overset{z}{0}) \quad \text{acts at CoM}$$

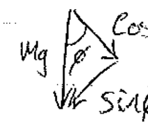
$$I\ddot{\phi} = -MgR\sin\phi \qquad (\text{from } \dot{J}_z = I\dot{\omega} = \sum \rho F_\phi)$$

Compare simple pendulum $\ddot{\phi} = -\frac{g}{l}\sin\phi$

$\Rightarrow \underbrace{\text{"}l\text{"}}_{\text{length of equivalent simple pendulum}} = \frac{I}{MR}$ frequency: $\omega = \sqrt{\frac{g}{l}} = \sqrt{\frac{gMR}{I}}$

Reaction $\vec{Q}$: $\dot{\vec{P}} = M\ddot{\vec{R}} = \vec{Q} + \sum \vec{F}$ (3)

$$\Rightarrow \quad \vec{Q} = M\ddot{\vec{R}} - \sum \vec{F}$$
$$= M\left(\dot{\vec{\omega}} \wedge \vec{R} + \vec{\omega} \wedge (\vec{\omega} \wedge \vec{R}) \right) - Mg\,\hat{x}$$



$Q_z = 0$ centripetal force provided by pivot

$Q_\rho = -MR\dot{\phi}^2 - Mg\cos\phi$

$Q_\phi = MR\ddot{\phi} + Mg\sin\phi$

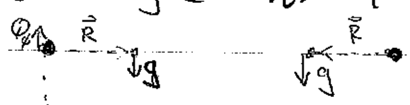
Re-write using $I\ddot{\phi} = -MgR\sin\phi$ and $E = \frac{1}{2}I\dot{\phi}^2 - MgR\cos\phi$

$$\left.\begin{aligned} Q_z &= 0 \\ Q_\rho &= -Mg\left(1 + 2R/l\right)\cos\phi - 2E/l \\ Q_\phi &= Mg\left(1 - R/l\right)\sin\phi \end{aligned}\right\}$$

purely a function of coords, not velocities and acceleration

"l" $= \frac{I}{MR}$

N.B. $l > R$



N.B. simple pendulum : $Q_z = 0$

$Q_\rho = -ml\dot{\phi}^2 - mg\cos\phi$

$Q_\phi = ml\ddot{\phi} + mg\sin\phi$

but $\ddot{\phi} = -\frac{g}{l}\sin\phi$ and $E = \frac{1}{2}ml^2\dot{\phi}^2 - mgl\cos\phi$

$$\Rightarrow ml\dot{\phi}^2 = \frac{2}{l}\left(E + mgl\cos\phi\right)$$

So $Q_z = 0$

$Q_\rho = -3mg\cos\phi - 2E/l$

$Q_\phi = -mg\sin\phi + mg\sin\phi = 0$ (simple pendulum point mass has no extension and so can't spin).

[end of not covered in class]

Perpendicular components of Ang. Mom.

So far we only considered J_z

cartesian coords: $\dot{x} = -\omega y$, $\dot{y} = \omega x$, $\dot{z} = 0$

$\vec{J} = \vec{r} \wedge \vec{p}$

$$J_x = \sum m(-z\dot{y}) = -\sum m x z \omega$$
$$J_y = \sum m(z\dot{x}) = -\sum m y z \omega$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ m\dot{x} & m\dot{y} & m\dot{z} \end{vmatrix} \quad m\dot{z} = 0$$

$$= \hat{i}(0 - zm\dot{y}) - \hat{j}(0 - m\dot{x}z) + \hat{k}(xm\dot{y} - ym\dot{x})$$

$\dot{\vec{r}} = \vec{\omega} \wedge \vec{r}$

$\dot{x} = (\vec{\omega} \wedge \vec{r})_x$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = \hat{i}(-\omega y) - \hat{j}(-\omega x) + \hat{k}\,0 = -\hat{i}\omega y + \hat{j}\omega x$$

So $J_x = I_{xz}\omega$ $\quad J_y = I_{yz}\omega$ $\quad J_z = I_{zz}\omega$

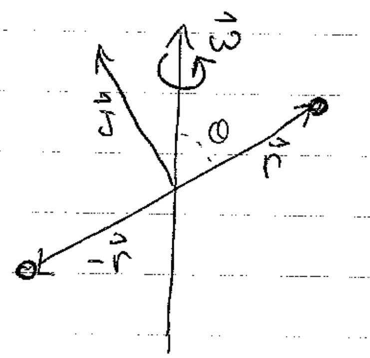
$$\underbrace{I_{xz} = -\sum m x z \; ; \quad I_{yz} = -\sum m y z}_{\text{products of inertia}} \qquad \underbrace{I_{zz} = \sum m(x^2+y^2)}_{\text{Moment of inertia about z-axis}}$$

previously denoted I.

Note: not constants! $\dot{I}_{xz} = -\omega I_{yz}$, $\dot{I}_{yz} = \omega I_{xz}$, $\dot{I}_{zz} = 0$

ie $\dot{I}_{xz} = -\sum m \dot{x} z$ but $\dot{x} = -\omega y$ ....

Example



2 masses connected by light rod, rigidly fixed at angle θ

$$\vec{J} = m\vec{r} \wedge \dot{\vec{r}} + m(-\vec{r}) \wedge (-\dot{\vec{r}}) = 2m\vec{r} \wedge (\vec{\omega} \wedge \vec{r})$$

$\vec{J}$ is $\perp$ to $\vec{r}$

(5)

When rod is in xz-plane, masses move in $\pm y$ direction
$\Rightarrow$ component of A.M. about x-axis and z-axis.

~~Note that~~ So $\vec{J}$ and $\vec{\omega}$ in general point in different directions!

Special case: products of inertia vanish, $\Rightarrow \vec{J}$ is in z-direction.
$\Rightarrow$ z-axis is called "principal axis of inertia".

e.g. z-axis is a principal axis when

- xy plane is a plane of reflection symmetry (contribution to I_{xz} and I_{yz} from any point (x, y, z) cancel with those from $(x, y, -z)$)

- If z-axis is an axis of rotational symmetry [(x,y,z) contribution cancelled by $(-x, -y, z)$]

For bodies with 3 symmetry axes (e.g. rectangular parallelepiped, ellipsoid) there are 3 ~~principle~~ perpendicular principal axes [in fact this is always true] $\Rightarrow$ advantageous to choose these as coord. axes
We shall no longer assume z-axis as axis of rotation.

$$\vec{\omega} = (\omega_x, \omega_y, \omega_z)$$

$$\vec{J} = \sum m \vec{r} \wedge (\vec{\omega} \wedge \vec{r}) = \underbrace{\sum m [r^2 \vec{\omega} - (\vec{r} \cdot \vec{\omega}) \vec{r}]}_{\text{linear combinations of components of } \vec{\omega}}$$

e.g. $J_z = \sum m [(x^2+y^2+z^2)\omega_z - (x\omega_x + y\omega_y + z\omega_z) z]$

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \underbrace{\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}}_{\text{inertia tensor } \overline{\overline{I}}} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

So, e.g. $J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$ etc. (2 other eqns) (6)

$\bar{\bar{I}}$ is symmetric ; e.g. $I_{xy} = I_{yx}$

If coordinate axes are all axes of symmetry

$$\bar{\bar{I}} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix} \quad \text{diagonal}$$

then : $J_x = I_{xx}\omega_x$, $J_y = I_{yy}\omega_y$, $J_z = I_{zz}\omega_z$

$\Rightarrow \vec{J}$ is parallel to $\vec{\omega}$ if $\vec{\omega}$ is along any one of the symmetry axes.

In fact, ~~for~~ any symmetric tensor can be diagonalised
$\Rightarrow$ any rigid body has three perpendicular axes through any given point which are principal axes of inertia $\{e_1, e_2, e_3\}$

then $\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$

unit vectors along principle axes

$\Rightarrow \vec{J} = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$

I_1, I_2, I_3 = principal moments of inertia (3 diagonal elements of $\bar{\bar{I}}$)
single subscript distinguishes them from moments of inertia about arbitrary axes.

Note : Principal axes are fixed in body and define a rotating frame.
Principal moments I_1, I_2, I_3 are then constants.

$$T = \sum \tfrac{1}{2} m \dot{r}^2 = \sum \tfrac{1}{2} m (\vec{\omega} \wedge \vec{r})^2 = \sum \tfrac{1}{2} m [\omega^2 r^2 - (\vec{\omega}.\vec{r})^2]$$ 7

(by vector algebra)

$$\Rightarrow \quad T = \tfrac{1}{2} \vec{\omega} . \vec{J}$$

$$\Rightarrow \quad T = \tfrac{1}{2} I_1 \omega_1^2 + \tfrac{1}{2} I_2 \omega_2^2 + \tfrac{1}{2} I_3 \omega_3^2$$

cf $\quad T = \tfrac{1}{2} M v^2 = \tfrac{1}{2} \vec{v}.\vec{p}$ — different "masses" in different directions.

Symmetric bodies

2 principal moments of inertia coincide, e.g. $I_1 = I_2$.
e.g. if $\hat{e}_3$ is a symmetry axis. (cylindrical symmetry or axis of more than 2-fold rotational symmetry, e.g. equilateral triangular prism, or square pyramid. - see book p 207

then $\quad \vec{J} = I_1 (\omega_1 \hat{e}_1 + \omega_2 \hat{e}_2) + I_3 \omega_3 \hat{e}_3$

If $\omega_3 = 0$, $\vec{J} = I_1 \vec{\omega}$, so $\vec{J}$ is parallel to $\vec{\omega}$.
Any axis in plane of $\hat{e}_1$ and $\hat{e}_2$ is a principal axis.
$\Rightarrow$ choose any pair of orthogonal axes in plane normal to $\hat{e}_3$.
$\Rightarrow$ they need not even be fixed in body providing they remain orthogonal to themselves and $\hat{e}_3$.

Totally symmetric : $I_1 = I_2 = I_3$ Sphere, cube, regular tetrahedron

$\vec{J} = I_1 \vec{\omega}$ whatever direction $\vec{\omega}$ takes, so every axis is a principal axis.

N.B. see sec 9.5 for calculation of Moments of inertia

Effect of a small force on axis

- Assume rigid body can rotate about a fixed pivot, i.e. only one pt on axis is fixed

- Assume initially body is freely rotating about a principal axis, $\hat{e}_3$.

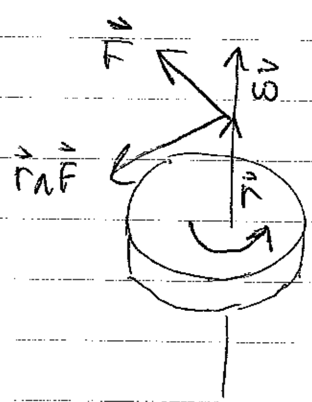
$$\Rightarrow \quad \vec{\omega} = \omega \hat{e}_3$$

$$\vec{J} = I_3 \omega \hat{e}_3$$

No external force : $\dot{\vec{J}} = I_3 \dot{\vec{\omega}} = 0$ [from $\vec{J} = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$]

$\Rightarrow$ axis fixed in space and $\vec{J}$ and $\vec{\omega}$ are constant (not true if axis of rotation was not a principal axis)

Apply small force $\vec{F}$ to axis at pt $\vec{r}$



Eqn of motion $\dot{\vec{J}} = \vec{r} \wedge \vec{F}$

$\Rightarrow$ ~~Axis~~ Axis changes direction

$\Rightarrow$ Small component of $\vec{\omega}$ perpendicular to $\hat{e}_3$

small F $\Rightarrow$ $|\delta\vec{\omega}|$ small in comparison to $|\vec{\omega}|$

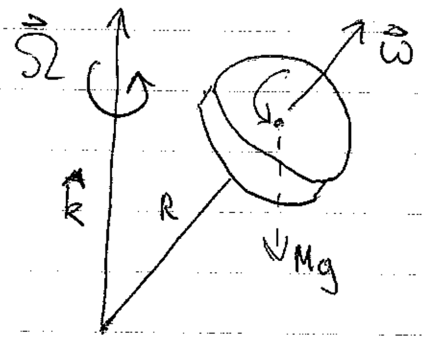
i.e. angular velocity with which axis moves is small in comparison to angular velocity about axis

$\Rightarrow$ ignore components of $\vec{J}$ normal to axis of rotation.

$$\Rightarrow \quad \dot{\vec{J}} = I_3 \dot{\vec{\omega}} = \underbrace{\vec{r} \wedge \vec{F}}_{\text{torque}}$$

$\vec{r} \wedge \vec{F}$ is $\perp$ to $\vec{\omega}$, so $|\vec{\omega}|$ does not change $\left(\frac{d\omega^2}{dt} = 2\vec{\omega} \cdot \dot{\vec{\omega}} \right)$ but direction of $\vec{\omega}$ does change. Axis will move in direction $\vec{r} \wedge \vec{F}$ which is $\perp$ to $\vec{F}$!

Example Precession of spinning top / gyroscope.



$\vec{F} = -Mg\hat{k}$ acts on COM
at $\vec{R} = R\hat{e}_3$

$$\therefore \quad I_3 \omega \dot{\hat{e}}_3 = -MgR\, \hat{e}_3 \wedge \hat{k} \qquad (I_3\dot{\vec{\omega}} = \vec{r} \wedge \vec{F})$$

cf. $\dot{\hat{e}}_3 = \vec{\Omega} \wedge \hat{e}_3$

$$\Rightarrow \quad \vec{\Omega} = \frac{MgR}{I_3 \omega}\hat{k}$$ Angular velocity of precession.

Note Ω is independent of angle of inclination

Treatment valid if $\Omega \ll \omega$

i.e. $MgR \ll I_3 \omega^2$

i.e. rotational KE $\gg$ change in gravitational potential
$T = \frac{1}{2} I_3 \omega^2$ $\qquad MgR$

Ω is small if I_3 and ω are big

Instantaneous Angular Velocity

For a body that is not rotating about a principal axis or not subjected to small forces when rotating about principal axis one ~~can~~ still define an instantaneous angular velocity but ~~[illegible]~~ in general both its magnitude and direction change in time.

First consider a body rotating freely about a fixed pivot.
$\Rightarrow$ position of every point of body is ~~fixed~~ if we specify it using the three principal axes $\vec{e}_1, \vec{e}_2, \vec{e}_3$ (actually true for any three axes fixed in body - but convenient choice is principal axes) $\vec{r} = r_1 \vec{e}_1 + r_2 \vec{e}_2 + r_3 \vec{e}_3$

Velocity of pt is determined by $\dot{\vec{e}}_1, \dot{\vec{e}}_2, \dot{\vec{e}}_3$

$$\dot{\vec{r}} = r_1 \dot{\vec{e}}_1 + r_2 \dot{\vec{e}}_2 + r_3 \dot{\vec{e}}_3$$

Can show that we can always define an instantaneous $\vec{\omega}$ no matter how body is moving:

Define $\quad a_{ij} = \vec{e}_i \cdot \dot{\vec{e}}_j$

$\vec{e}_1 \cdot \vec{e}_1 = 1 \quad$ etc

so $\quad 0 = \frac{d}{dt}(\vec{e}_1 \cdot \vec{e}_1) = 2\,\vec{e}_1 \cdot \dot{\vec{e}}_1 = 2a_{11}$

So $\dot{\vec{e}}_1$ must have form

$$\dot{\vec{e}}_1 = a_{21}\vec{e}_2 + a_{31}\vec{e}_3$$

[Check $\quad \vec{e}_2 \cdot \dot{\vec{e}}_1 = a_{21} \quad$ ~~and~~ and $\quad \vec{e}_3 \cdot \dot{\vec{e}}_1 = a_{31}$]

Also $\quad 0 = \frac{d}{dt}(\vec{e}_1 \cdot \vec{e}_2) = \dot{\vec{e}}_1 \cdot \vec{e}_2 + \vec{e}_1 \cdot \dot{\vec{e}}_2 = a_{21} + a_{12}$

So can define a vector $\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$

where
$$\omega_1 = -a_{23} = a_{32}$$
$$\omega_2 = -a_{31} = a_{13}$$
$$\omega_3 = -a_{12} = a_{21}$$

for then
$$\dot{\vec{e}}_1 = \cancel{\omega_3 \vec{e}_2 - \omega_2 \vec{e}_3} = a_{21} \vec{e}_2 + a_{31} \vec{e}_3$$
$$= \omega_3 \hat{e}_2 - \omega_2 \hat{e}_3$$
$$= \vec{\omega} \wedge \vec{e}_1$$

$$\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ 1 & 0 & 0 \end{vmatrix} = \hat{e}_1(0) - \hat{e}_2(-\omega_3) + \hat{e}_3(-\omega_2)$$ as req'd

Combining with
$$\dot{\hat{e}}_2 = \vec{\omega} \wedge \vec{e}_2$$
$$\dot{\hat{e}}_3 = \vec{\omega} \wedge \vec{e}_3$$

gives $\boxed{\dot{\vec{r}} = \vec{\omega} \wedge \vec{r}}$ $\left[\dot{\vec{r}} = r_1 \dot{\vec{e}}_1 + r_2 \dot{\vec{e}}_2 + r_3 \dot{\vec{e}}_3\right]$

(velocity of any pt in body) sub in $\dot{\vec{e}}_1 = \vec{\omega} \wedge \vec{e}_1$

$$\dot{\vec{r}} = r_1\, \vec{\omega} \wedge \hat{e}_1 + r_2\, \vec{\omega} \wedge \hat{e}_2 + r_3\, \vec{\omega} \wedge \hat{e}_3$$
$$= \vec{\omega} \wedge (r_1 \hat{e}_1 + r_2 \hat{e}_2 + r_3 \hat{e}_3)$$
$$= \vec{\omega} \wedge \vec{r}$$

So instantaneous angular velocity vector $\vec{\omega}$ always exists. (instantaneous axis of rotation)

<u>Now consider general case</u>: no pt in body is fixed.

Specify position of body by $\vec{R}$ (com position) and by orientations of principal axes at $\vec{R}$. Then define $\vec{\omega}$ relative to $\dot{\vec{R}}$.

$\dot{\vec{R}}$ = velocity of COM

velocity of pt : $\dot{\vec{r}} = \dot{\vec{R}} + \dot{\vec{r}}^*$

position relative to COM

But $\vec{r}^*$ is fixed in body so can same argument as above to define instantaneous angular velocity $\vec{\omega}$

$$\dot{\vec{r}} = \dot{\vec{R}} + \vec{\omega} \wedge \vec{r}^*$$

no star because $\vec{\omega}$ is independent of origin of coord

e.g. if body is rotating with $\vec{\omega}$ about a fixed pivot:

$$\dot{\vec{r}} = \vec{\omega} \wedge \vec{r} = \vec{\omega} \wedge \vec{R} + \vec{\omega} \wedge \vec{r}^*$$

but $\dot{\vec{R}} = \vec{\omega} \wedge \vec{R}$

so $\underbrace{\dot{\vec{r}} - \dot{\vec{R}}}_{= \dot{\vec{r}}^*} = \vec{\omega} \wedge \vec{r}^*$

so $\vec{\omega}$ is same also relative to COM.

Equ of motion for rigid body about fixed pivot:

$$\dot{\vec{J}} = \sum \vec{r} \wedge \vec{F}$$

where $\vec{J} = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$

In general case we need to know both velocity of COM and $\vec{\omega}$

Motion of COM

$$\dot{\vec{P}} = M\ddot{\vec{R}} = \sum \vec{F}$$

Rotational motion

$$\dot{\vec{J}}^* = \sum \vec{r}^* \wedge \vec{F}$$

$\vec{J}^*$ = ang. mom. about COM

$$\vec{J}^* = I_1^* \omega_1 \hat{e}_1^* + I_2^* \omega_2 \hat{e}_2^* + I_3^* \omega_3 \hat{e}_3^*$$

$\vec{e}_1^*, \vec{e}_2^*, \vec{e}_3^*$ = principal axes at centre of mass
and I_1^*, I_2^*, I_3^* are corresponding principal moments.

Rotation about a principal Axis

Principal axes $\hat{e}_1, \hat{e}_2, \hat{e}_3$ rotate with body $\Rightarrow$ rotating frame.

$\frac{d\vec{J}}{dt}$ = ~~relative~~ absolute

$\dot{\vec{J}}$ = relative

Absolute $\frac{d\vec{J}}{dt} = \sum \vec{r} \wedge \vec{F} = \vec{\tau}$ (argument below applies equally to body rotating about fixed pivot or COM.)

relative $\dot{\vec{J}} = I_1 \dot{\omega}_1 \hat{e}_1 + I_2 \dot{\omega}_2 \hat{e}_2 + I_3 \dot{\omega}_3 \hat{e}_3$ (I_1, I_2, I_3 = consts)

relation : $\frac{d\vec{J}}{dt} = \underbrace{\dot{\vec{J}}}_{\text{rate of change in frame}} + \underbrace{\vec{\omega} \wedge \vec{J}}_{\text{rate of change of } \vec{J} \text{ due to rotation of frame}} = \vec{\tau}$

In terms of components: (The Euler equations)

$$I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = \tau_1$$

$$I_2 \dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_2 = \tau_2$$

$$I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = \tau_3$$

Can solve for $\omega_1(t), \omega_2(t), \omega_3(t)$
- but not so useful if external forces are specified in terms of components wrt a fixed set of axes.
(Even if force is const, $\{F_1, F_2, F_3\}$ depend on time, i.e. orientation of body) ~~Which we can only know by solving~~

... consider free rotation in example below (use Lagrange's eqns for $\vec{\tau} \neq 0$.)

Tennis ~~racket~~ racquet theorem

Rigid body is rotating freely about principal axis, $\hat{e}_3$, say.
Give small displacement – will $\vec{\omega}$ remain close to $\hat{e}_3$?

If displacement is small, treat ω_1 and ω_2 as small and $\omega_1 \omega_2 \simeq 0$.

then 3rd eqn $\quad I_3 \dot{\omega}_3 \simeq 0 \qquad (\tau_3 = 0)$

$$\Rightarrow \omega_3 = \text{const.}$$

Solve other two eqns by putting $\omega_1 = a_1 e^{pt}$, $\omega_2 = a_2 e^{pt}$

$$\Rightarrow \quad I_1 p a_1 + (I_3 - I_2)\omega_3 a_2 = 0$$
$$I_2 p a_2 + (I_1 - I_3)\omega_3 a_1 = 0$$

eliminate a_1/a_2

$$\Rightarrow \quad p^2 = \frac{(I_3 - I_2)(I_1 - I_3)}{I_1 I_2}\,\omega_3^2$$

If $I_3 > I_1$ and $I_3 > I_2$
or $I_3 < I_1$ and $I_3 < I_2$
$\Big\}$ p = imaginary $\Rightarrow$ oscillatory solutions

$I_1 > I_3 > I_2$
or $I_1 < I_3 < I_2$
$\Big\}$ real roots $\Rightarrow$ ω_1 and ω_2 exponentially increasing.

$\Rightarrow$ motion is stable if I_3 is largest ~~or~~ or smallest principal moment.

Euler's Angles (θ, ϕ, ψ)

$\hat{e}_1, \hat{e}_2, \hat{e}_3$ initially coincide with $\hat{i}, \hat{j}, \hat{k}$

① Rotate about axis $\hat{k}$ by angle ϕ
$\Rightarrow \hat{e}_1'', \hat{e}_2', \hat{k}$

② Rotate about the axis $\hat{e}_2'$ by angle θ
$\Rightarrow \hat{e}_1', \hat{e}_2', \hat{e}_3$

③ Rotate about $\hat{e}_3$ by angle ψ
$\Rightarrow \hat{e}_1, \hat{e}_2, \hat{e}_3$

Angular velocity

Summary

Rotation About axis

$$\vec{J} = \sum m \vec{r} \wedge (\vec{\omega} \wedge \vec{r}) = \sum m \left[r^2 \vec{\omega} - (\vec{r}.\vec{\omega})\vec{r} \right]$$

linear in $\vec{\omega}$.

Cartesian coord

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \underbrace{\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}}_{\bar{\bar{I}} \text{ symmetric}} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

e.g. $J_x = I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z$

If coord axes are axes of symmetry

$$\bar{\bar{I}} = \begin{bmatrix} I_{xx} & & \\ & I_{yy} & \\ & & I_{zz} \end{bmatrix}$$

$J_x = I_{xx}\,\omega_x \qquad J_y = I_{yy}\,\omega_y \qquad J_z = I_{zz}\,\omega_z$

$\Rightarrow \vec{J}$ parallel to $\vec{\omega}$ if $\vec{\omega}$ is along any one symmetry axis

If coord axes are not axes of symmetry : diagonalise $\Rightarrow \hat{e}_1, \hat{e}_2, \hat{e}_3$

Then can write $\vec{\omega} = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$ this in basis

then $J = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$

principal moments of inertia