Poincaré - Birkhoff fixed point theorem

Can now tackle the fundamental question of the fate of tori with rational frequency ratios under small perturbation.

$(i+1)^{th}$ iterate i^{th} iterate

Use twist map:
$$\phi' = \phi + \frac{\partial}{\partial I'} S_0(I')$$
$$I' = I \qquad \text{(unperturbed)}$$

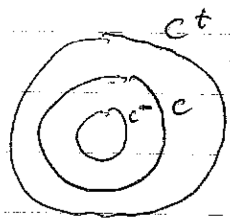
2D tori:
$$\frac{\partial S_0}{\partial I} = 2\pi \frac{\omega_1}{\omega_2}$$

Perturbed twist map:
$$\phi' = \phi + \frac{\partial}{\partial I'} S_0(I') + \epsilon \frac{\partial}{\partial I'} S_1(I', \phi)$$
$$I' = I + \epsilon \frac{\partial}{\partial \phi} S_1(I, \phi)$$

KAM: "most" invariant curves (tori) will be preserved providing
$$\det \left| \frac{\partial^2 S_0}{\partial I_i \partial I_j} \right| \neq 0 \qquad \text{(non-degeneracy)}$$

with the exception of rational curves: $\alpha = \frac{\omega_1}{\omega_2} = \frac{r}{s}$

Consider two curves C^+ and C^- which lie on either side of C which has $\alpha = \frac{r}{s}$.



C, C^+, C^- are invariant curves of T.

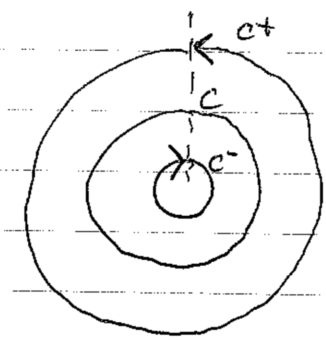
$$\begin{bmatrix} \phi' \\ I' \end{bmatrix} = T \begin{bmatrix} \phi \\ I \end{bmatrix}$$

Every pt on C is a fixed pt of T^s

$$T^s \begin{bmatrix} \phi \\ I \end{bmatrix} = \begin{bmatrix} \phi + s\left(\frac{\partial S_0}{\partial I}\right) \\ I \end{bmatrix} = \begin{bmatrix} \phi + s\, 2\pi \left(\frac{r}{s}\right) \\ I \end{bmatrix}$$

$$= \begin{bmatrix} \phi + 2\pi r \\ I \end{bmatrix} = \begin{bmatrix} \phi \\ I \end{bmatrix}$$

Relative to C, C^+ rotates anticlockwise and C^- rotates clockwise under T^s



$\alpha > \frac{r}{s}$ for C^+

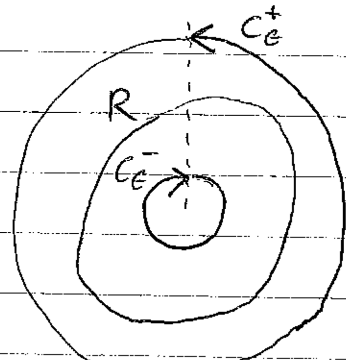
$\alpha < \frac{r}{s}$ for C^-

Perturbed twist map T_ϵ : by KAM C^+ and C^- are preserved, albeit in a slightly distorted form C_ϵ^+, C_ϵ^-. These curves are invariant under T_ϵ.

If ϵ is small the relative twists of C_ϵ^+ and C_ϵ^- are preserved under T_ϵ^s.

$\Rightarrow$ along each radius vector ~~emana~~ emanating from centre there is one point whose angular coordinate ϕ is preserved. Draw curve R through these pts

by continuity

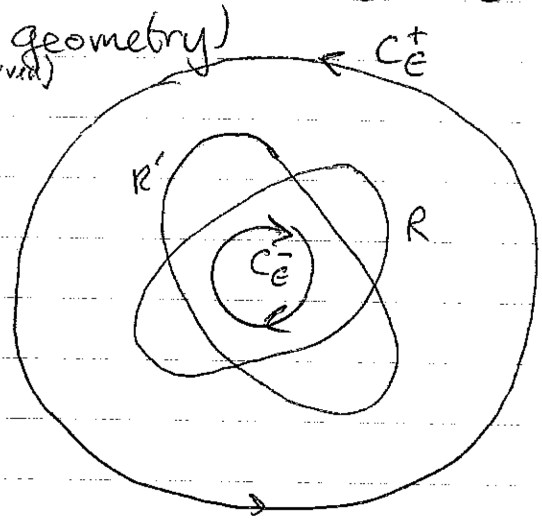


R is <u>not</u> invariant under T_ϵ. Neither is it invariant under T_ϵ^s <u>but</u> a subset of points on it are the fixed pts of T_ϵ^s i.e. a fixed pt has fixed angle and radius : R is just the curve of preserved angles.

Can find the fixed pts of T_ϵ^s by applying T_ϵ^s to R

$$R' = T_\epsilon^s R$$

R' must intersect R at an even number of pts (by simple geometry) (e.g. area is preserved) - these are the fixed pts of T_ϵ^s



Poincaré - Birkhoff fixed pt thm: for a rational curve of an unperturbed system with $\alpha = \frac{r}{s}$ (for which every pt is a fixed pt of T^s) only an even number of fixed pts $2ks$ $(k = 1, 2, 3 \ldots)$ will remain under perturbation.

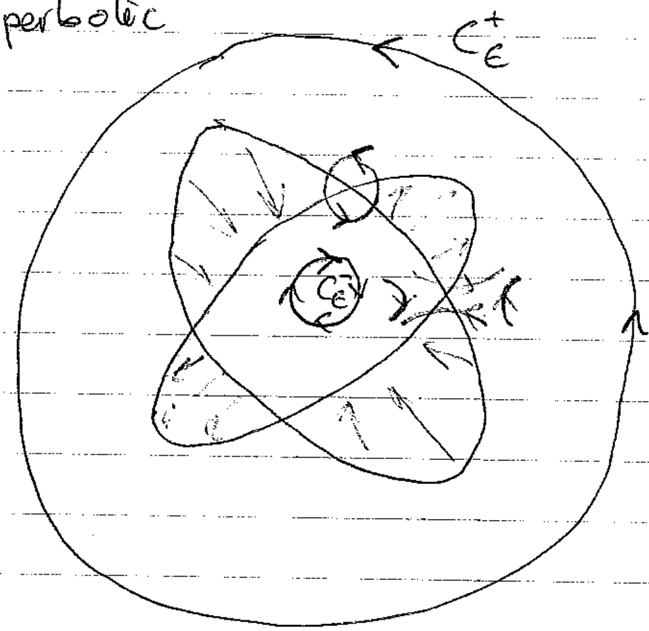
Why an even multiple of s? Take one of the F.P.s given by intersection of R and R'. By definition it is a F.P. of T_ϵ^s.

Under T_ϵ its orbit is

$$X,\ T_\epsilon X,\ T_\epsilon^2 X, \ldots T_\epsilon^{s-1} X \qquad \text{(a closed orbit)}$$

However, each of those pts is a F.P. of T_ϵ^s
$\Rightarrow$ s ~~fixed~~ F.P.'s associated with each intersection of ~~R~~ R and R'.

<u>Note</u>: F.P.s of T_ϵ^s are alternately elliptic and hyperbolic



follow the flow lines...

$\Rightarrow$ ks are elliptic

ks are hyperbolic

Around each elliptic F.P. we find a family of invariant curves. These are also subject to KAM thm $\Rightarrow$ rational members break up ~~with~~ according to P-B. thm $\Rightarrow$ same structure is repeated around this subsequence of elliptic F.P. $\Rightarrow$ self-similar structure on all scales.