Orbits

We want $r(\theta)$. Chpt 4

Eliminate time from:

Conservation laws $\begin{cases} \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = E \\ m r^2 \dot{\theta} = J \end{cases}$

Combine: $\frac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) = E$

Define $u = \frac{1}{r}$. Work with this instead $u(\theta)$?

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$$

$$\Rightarrow \quad \dot{r} = \frac{dr}{d\theta}\dot{\theta} = -r^2\dot{\theta}\frac{du}{d\theta} = -\frac{J}{m}\frac{du}{d\theta}$$

sub into radial equation

$$\frac{J^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{J^2}{2m}u^2 + V(u) = E$$

$V(u) = ku$ for inverse square law

As before, define length $l = \frac{J^2}{m|k|}$ $\Rightarrow$ $\frac{|k|}{2} l \left(\frac{du}{d\theta}\right)^2 + \frac{|k|}{2} l u^2 \pm ku = E$

Mult. radial eqn by $\frac{2}{|k|}$

$$l\left(\frac{du}{d\theta}\right)^2 + lu^2 \pm 2u = \frac{2E}{|k|}$$

$+ : k > 0$

$- : k < 0$

To solve, mult. by l and add 1

$$l^2\left(\frac{du}{d\theta}\right)^2 + l^2u^2 \pm 2ul + 1 = \frac{2El}{|k|} + 1$$

but $\quad l^2u^2 \pm 2ul + 1 = (lu \pm 1)^2$

Define

$$Z \equiv lu \pm 1 \qquad \text{then} \quad \frac{dz}{d\theta} = l\frac{du}{d\theta}$$

and

$$\left(\frac{dz}{d\theta}\right)^2 + Z^2 = \frac{2El}{|k|} + 1 \equiv e^2$$

dimensionless constant.

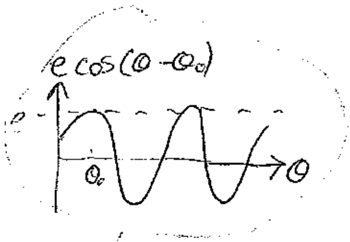
N.B. only has sol^n when $\frac{2El}{|k|} + 1 > 0$

i.e. $\quad E > \frac{-|k|}{2l}$

General solution of equation: $\quad Z = e\cos(\theta - \theta_0)$

const. of integration

but $\quad Z = l\frac{1}{r} \pm 1$, so



$$r[e\cos(\theta-\theta_0) - 1] = l \qquad k>0 \qquad \text{repulsive}$$

$$r[e\cos(\theta-\theta_0) + 1] = l \qquad k<0 \qquad \text{attractive}$$

$\Rightarrow$ Conic sections with one focus as origin

$e =$ eccentricity

$l =$ semi-latus rectum

$\theta_0 =$ orientation of l w.r.t. coordinate axes

N.B. repulsive case: e must be > 1 (else [ ] always -ve). But $e^2 = \frac{2El}{|k|} + 1$ so $E > 0$

attractive case: when $e \geq 1 \Rightarrow E \geq 0$, then $r \to \infty$ when $[\ ] = 0$

i.e. particle escapes if $E > 0$

For both cases:

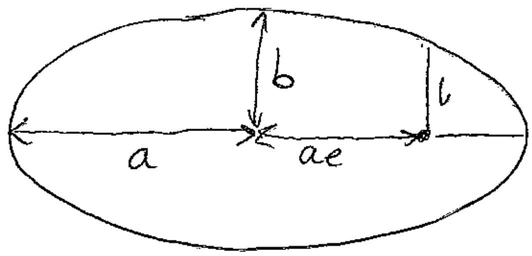
r takes its min value when $\theta = \theta_0$

$\Rightarrow \theta_0$ specifies direction of pt of closest approach

In attractive case l has simple geometric meaning: $r = l$ when $\theta = \theta_0 \pm \frac{\pi}{2}$

$\Rightarrow$ l is radial distance at right angles to closest approach

Elliptic orbits $(E < 0, 0 \leq e < 1)$



Re-write orbit equation in cartesian form, chose $\theta_0 = 0$

$$r(e\cos\theta \underbrace{\pm}_{\text{attractive / repulsive}} 1) = l \qquad \begin{array}{l} e \geq 0 \\ l > 0 \end{array}$$

Write as $\underbrace{re\cos\theta}_{ex} \pm r = l$

Or $\pm r = l - ex$

Square: $r^2 = x^2 + y^2 = (l - ex)^2$

or $x^2 - e^2x^2 + 2lex + y^2 = l^2$

$$\Rightarrow (1-e^2)x^2 + 2elx + \frac{e^2l^2}{1-e^2} - \frac{e^2l^2}{1-e^2} + y^2 = l^2$$

$$\Rightarrow \left(\sqrt{1-e^2}\,x + \frac{el}{\sqrt{1-e^2}}\right)^2 + y^2 = l^2 + \frac{e^2l^2}{1-e^2} = l^2\left(1 + \frac{e^2}{1-e^2}\right)$$

$$\left(\sqrt{1-e^2}\,x + \frac{el}{\sqrt{1-e^2}}\right)^2 + y^2 = \frac{l^2}{1-e^2}$$

or $$(1-e^2)\left(x + \frac{el}{1-e^2}\right)^2 + y^2 = \frac{l^2}{1-e^2}$$

$$\Rightarrow \quad \frac{(1-e^2)^2}{l^2}\left(x + \frac{el}{1-e^2}\right)^2 + y^2\frac{(1-e^2)}{l^2} = 1$$

Define $a \equiv \dfrac{l}{1-e^2}$, $\quad b \equiv \dfrac{l}{\sqrt{1-e^2}}$ semi-axes

then $$\frac{(x+ea)^2}{a^2} + \frac{y^2}{b^2} = 1$$

same as $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ but with origin at $(-ae, 0)$.

N.B. $e^2 = \dfrac{a^2-b^2}{a^2}$ $\qquad l = \dfrac{b^2}{a}$

Physically $a = \dfrac{l}{1-e^2} = \dfrac{l}{1-\left(1+\frac{2El}{|k|}\right)} = \dfrac{|k|}{-2E} = \dfrac{|k|}{2|E|}$ semi-axis a is determined by energy E.

$b^2 = al = \dfrac{|k|}{2|E|}\dfrac{J^2}{m|k|} = \dfrac{J^2}{2m|E|}$ So l is fixed by J.

Time taken to traverse any part of orbit is found from

$$\frac{dA}{dt} = \frac{J}{2m}$$

$$\Rightarrow \quad dt = dA \, \frac{2m}{J}$$

area swept out by radius vector

Period of orbit? $A = \pi a b$

so $$\tau = \pi a b \, \frac{2m}{J}$$

or $$\left(\frac{\tau}{2\pi}\right)^2 = \frac{m^2 a^2 b^2}{J^2} = \frac{m^2 a^3 l}{J^2}$$

but $l = \frac{J^2}{m|k|}$ so $$\left(\frac{\tau}{2\pi}\right)^2 = \frac{m^2 a^3}{m|k|} = \frac{m}{|k|} a^3$$

So for gravity $k = GM$ and $$\left(\frac{\tau}{2\pi}\right)^2 = \frac{a^3}{GM}$$

Kepler's 3rd law.

Hyperbolic orbits $(E>0,\ e>1)$

Again $x^2+y^2 = (l - ex)^2$

Complete square as before:

$$(1-e^2)x^2 + 2elx + \frac{e^2l^2}{1-e^2} - \frac{e^2l^2}{1-e^2} + y^2 = l^2$$

multi. by -1:

$$(e^2-1)x^2 - 2elx + \frac{e^2l^2}{e^2-1} - \frac{e^2l^2}{e^2-1} - y^2 = -l^2$$

$$\left(\sqrt{e^2-1}\,x - \frac{el}{\sqrt{e^2-1}}\right)^2 - y^2 = \frac{e^2l^2}{e^2-1} - l^2$$

$$= l^2\left(\frac{e^2}{e^2-1} - 1\right) = \frac{l^2}{e^2-1}$$

$$\Rightarrow (e^2-1)\left(x - \frac{el}{e^2-1}\right)^2 - y^2 = \frac{l^2}{e^2-1}$$

$$\Rightarrow \frac{(e^2-1)^2}{l^2}\left(x - \frac{el}{e^2-1}\right)^2 - y^2\frac{(e^2-1)}{l^2} = 1$$

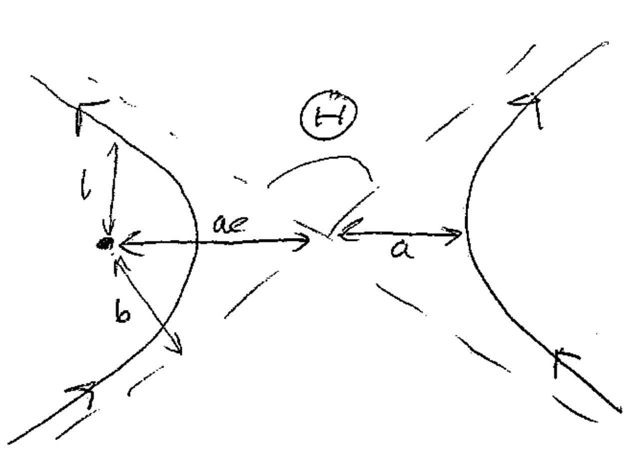
Define $a = \dfrac{l}{e^2-1}$, $b = \dfrac{l}{\sqrt{e^2-1}}$

$$\frac{(x-ea)^2}{a^2} - \frac{y^2}{b^2} = 1$$

$\Rightarrow$ hyperbola with centre at $(ae, 0)$

left hand branch = orbit under attractive $1/r$ force (intersects x axis at $(ae-a, 0)$

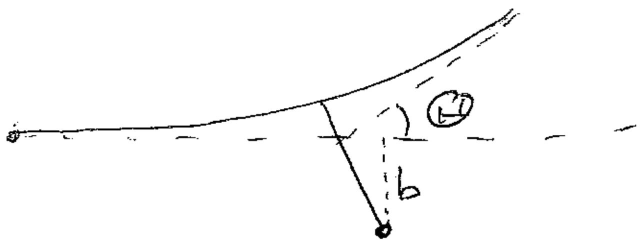
right " " = " " repulsive (" " " " $(ae+a, 0)$



note that $a = \frac{l}{e^2-1} = \frac{|k|}{2E}$ $\qquad b^2 = a l = \frac{J^2}{2mE}$

a is determined by energy
l " " " J

Also semiminor axis b is identical to impact parameter



<u>Asymptotes</u>:
i.e. $r \to \infty$
i.e. $(e\cos\theta \pm 1) = 0$
from $r(e\cos\theta \pm 1) = l$ (attractive / rep)

$$\theta = \pm \arccos\left(\tfrac{1}{e}\right) \qquad \text{repulsive}$$

$$\theta = \pm\left[\pi - \arccos\left(\tfrac{1}{e}\right)\right] \qquad \text{attractive}$$

<u>Deflection</u> angle in both cases:
scattering angle

$$\Theta = \pi - 2\arccos\left(\tfrac{1}{e}\right)$$

What is relation between impact parameter b, ~~and~~ scattering angle Θ, and limiting velocity v at infinity?

from $\Theta = \pi - 2\arccos(1/e)$

$$\Rightarrow e = \sec\left[\tfrac{1}{2}(\pi - \Theta)\right] \qquad \sec = \frac{1}{\cos}$$

sub in $b^2 = al = \dfrac{l^2}{e^2-1} = a^2(e^2-1)$ $\qquad \left[a = \dfrac{l}{e^2-1}\right]$

$$= a^2\left[\sec^2\left[\tfrac{1}{2}(\pi - \Theta)\right] - 1\right] = a^2\left[\frac{1}{\cos^2} - \frac{\cos^2}{\cos^2}\right]$$

$$= a^2 \cot^2 \frac{\Theta}{2}$$

$$= a^2 \frac{1-\cos^2}{\cos^2} = a^2 \frac{\sin^2\left(\frac{\pi-\Theta}{2}\right)}{\cos^2\left(\frac{\pi-\Theta}{2}\right)}$$

$$= a^2 \frac{\cos^2(\Theta/2)}{\sin^2(\Theta/2)}$$

but also, $a = \dfrac{|k|}{2E} = \dfrac{|k|}{2\,\frac{1}{2}mv^2} = \dfrac{|k|}{mv^2}$

$$\Rightarrow \quad b = \frac{|k|}{mv^2} \cos \frac{\Theta}{2}$$