Lagrangian formulation of mechanics

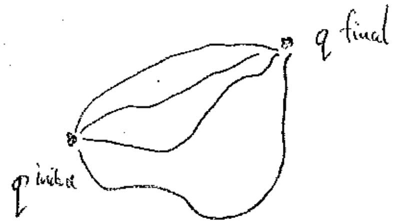
Newton's laws can be restated in a very elegant form using minimum principles like Fermat's principle of least time. Furthermore, this approach is very convenient because it introduces the idea of "generalized coordinates" q_i $(i = 1, \dots n)$ and "generalized velocities" $\dot{q}_i$ $(i = 1, \dots n)$ – N.B. these need not be cartesian coords

[don't need to explicitly include holonomic constraints]

Note that $x_i = x_i(q_1, \dots q_n, t)$ (cartesian coord) and $dx_i = \sum_j \frac{\partial x_i}{\partial q_j} dq_j + \frac{\partial x_i}{\partial t} dt$; $\frac{dx_i}{dt} = \sum_j \frac{\partial x_i}{\partial q_j} \frac{dq_j}{dt} + \frac{\partial x_i}{\partial t}$ → $\dot{x}_i = \dot{x}_i[q, \dot{q}, t]$

If system moves from $\vec{q}^{\,\text{initial}} = \{q_1(t_i) \dots q_n(t_i)\}$ at t_i ← initial

to $\vec{q}^{\,\text{final}} = \{q_1(t_f), \dots q_n(t_f)\}$ at t_f ← final.



find motion from Hamilton's principle of least action ; integral of the Lagrangian takes minimum possible value between t_i and t_f.

"Experience shows" that the state of system is completely described by coordinates + velocities. Then

$$L = L(q_1, \dots q_n ; \dot{q}_1, \dots \dot{q}_n ; t)$$

Hamilton's principle : Action integral $S = \int_{t_i}^{t_f} L(\vec{q}, \dot{\vec{q}}, t)\, dt$ is minimum.

Assume end points $\vec{q}^{\,\text{initial}}$ and $\vec{q}^{\,\text{final}}$ are fixed : $\delta q(t_i) = \delta q(t_f) = 0$

Vary :

$$\delta S = \int_{t_i}^{t_f} L(q_1 + \delta q_1, \dots q_n + \delta q_n ; \dot{q}_1 + \delta \dot{q}_1, \dots \dot{q}_n + \delta \dot{q}_n ; t) - \int_{t_i}^{t_f} L(q_1, \dots q_n ; \dot{q}_1, \dots \dot{q}_n ; t)\, dt$$

Expand L to first order

$$\delta S = \sum_j \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q_j} \delta q_j + \frac{\partial L}{\partial \dot{q}_j} \delta \dot{q}_j \right) dt$$

Put $\delta \dot{q} = \frac{d}{dt} \delta q$ and integrate 2nd term by parts

i.e. $$\int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{q}} \delta \dot{q}\, dt = \int_{t_i}^{t_f} \frac{\partial L}{\partial \dot{q}} \frac{d\delta q}{dt} dt$$

$$= \underbrace{\frac{\partial L}{\partial \dot{q}} \delta q \Big|_{t_i}^{t_f}}_{=0} - \int_{t_i}^{t_f} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \delta q\, dt$$

$$\delta S = \sum_j \int_{t_i}^{t_f} \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\right) \delta q_j\, dt = 0 \quad \text{(minimum)}$$

but this must hold for all δq_j

$$\Rightarrow \quad \frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \qquad \text{Lagrange's equation}$$

for Cartesian coords these are same as Newton's eqns.

put $L = T - V$, $F_j = -\frac{\partial}{\partial q_j} V$, $T = \sum_j \frac{1}{2} m \dot{q}_j^2$

For velocity independent potentials we get

$$m \ddot{q}_j = -\frac{\partial V}{\partial q_j}$$

Define generalized momenta

$$P_j = \frac{\partial L}{\partial \dot{q}_j}$$

generalized forces

$$F_j = \frac{\partial L}{\partial q_j}$$

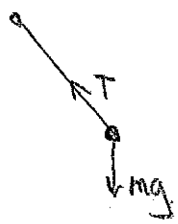
Note that p_i and F_i are <u>not</u> ~~in gen~~ always components of momentum vector $\vec{p}$ and force vector $\vec{F}$

Example: <u>Simple pendulum</u>

$$L = T - V = \tfrac{1}{2} m l^2 \dot{\theta}^2 - mgl(1-\cos\theta)$$

$$\frac{\partial L}{\partial \dot{\theta}} = m l^2 \dot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mgl \sin\theta$$

Lagrange eqn:



$$-mgl\sin\theta = m l^2 \ddot{\theta}$$

Have not needed to find expressions for polar components of acceleration as we would if we used Newton's laws. Also, have not needed to introduce tension in the rod.

Consider cylindrical polar coords ρ, ϕ, z : $T = \tfrac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$

(depends on time derivatives AND the coords themselves)

$$\frac{\partial T}{\partial \dot{\rho}} = m\dot{\rho} \qquad \frac{\partial T}{\partial \dot{\phi}} = m\rho^2\dot{\phi} \qquad \frac{\partial T}{\partial \dot{z}} = m\dot{z}$$

$$\frac{\partial T}{\partial \rho} = m\rho\dot{\phi}^2 \qquad \frac{\partial T}{\partial \phi} = 0 \qquad \frac{\partial T}{\partial z} = 0$$

Lagrange:

$$\frac{d}{dt}(m\dot{\rho}) = m\rho\dot{\phi}^2 - \frac{\partial V}{\partial \rho}$$

$$\frac{d}{dt}(m\rho^2\dot{\phi}) = -\frac{\partial V}{\partial \phi}$$

$$\frac{d}{dt}(m\dot{z}) = -\frac{\partial V}{\partial z}$$

Generalized momenta $m\dot{\rho}$ and $m\dot{z}$ are components of momentum vector $\vec{p}$ in the ρ and z directions.

But generalized momentum $m\rho^2\dot{\phi}$ is the angular momentum (not linear) J_z about z-axis.

Beware of notation P_ϕ = does it mean ϕ-component of $\vec{p}$ or the generalized momentum J_z ?

$\frac{dP_\phi}{dt} = F_\phi$

Only for cartesian coords do the Lagrange equation coincide with Newton's equations ~~[illegible]~~ i.e. only then do we obtain $P_j = m\dot{q}_j$,

Lagrange eqns give $\dot{P}_j = F_j$

> Generalized coords do not usually divide into convenient groups of three that can be associated together to form vectors (Goldstein, p 13) i.e. rotate like vectors

More general formulation to include arbitrary forces i.e. not necessarily of form $\vec{F} = -\nabla V$

Definition: A conservative force is one that depends only on position (not the path). Typically, NC forces depend on velocity.

This time we vary $I = \int_{t_i}^{t_f} T\,dt$ where $T = \frac{1}{2} m \dot{\vec{r}}^2$

$\frac{1}{2} m \left(\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 \right)$

Vary x-coordinate : $\delta x(t)$ subject to $\delta x(t_i) = \delta x(t_f) = 0$

$\Rightarrow \quad \delta T = m\dot{x}\,\delta\dot{x}$

Integrate by parts

$$\delta I = \underbrace{-\int_{t_i}^{t_f} m\ddot{x}\,\delta x\,dt}$$ (boundary terms vanish)

but $-F_x\,\delta x = -\delta W$ δW = work done by force $\vec{F}$ in displacement δx

Vary all 3 coords

$$\delta I = -\int_{t_i}^{t_f} \delta W \, dt$$

$$\delta W = \vec{F} \cdot \delta \vec{r} \quad \text{in the usual cartesian coords}$$

Just like Hamilton's principle, and in contrast to Newton's equs, there is no explicit reference to any particular set of coordinates. Can always write:

$$\delta W = F_1 \delta q_1 + F_2 \delta q_2 + F_3 \delta q_3 \qquad F_i = \text{generalised forces.}$$

~~then~~ However can also vary T using ~~standard~~ ~~the~~ previous method

$$\delta I = \sum_j \int_{t_i}^{t_f} \left[\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) \right] \delta q_j(t) \, dt$$

Note: here we vary T wrt q and $\dot{q}$ as opposed to cartesian coords and velocities. (see note below)

equate:

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) = \frac{\partial T}{\partial q_j} + F_j}$$

If force is conservative, then $\delta W = -\delta V(q_1, \dots q_n)$ ($\delta W = \delta T$ but $T + U = \text{const}$, $\delta T = -\delta V$)

$$\text{so} \quad F_j = -\frac{\partial V}{\partial q_j}$$

(using $\delta W = F_1 \delta q_1 + F_2 \delta q_2 + F_3 \delta q_3$)

then define $L = T - V \Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$

Note $T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$ cartesian coords.

now, $\vec{r} = \vec{r}(q_1, \dots q_n, t)$ (position of particle) $\Rightarrow dr_i = \sum_j \frac{\partial r_i}{\partial q_j} dq_j + \frac{\partial r_i}{\partial t} dt \Rightarrow \frac{dr_i}{dt} = \sum_j \frac{\partial r_i}{\partial q_j} \frac{\partial q_j}{\partial t} + \frac{\partial r_i}{\partial t}$

so $\dot{\vec{r}} = \dot{\vec{r}}(q_1, \dots q_n; \dot{q}_1, \dots \dot{q}_n; t)$, so we can have $\frac{\partial L}{\partial q_j} \neq 0$ and $\frac{\partial L}{\partial \dot{q}_j} \neq 0$.

Example

2D Pendulum

Assume that we also apply torque $\tau(t)$ for some time

$$L = \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta\right) - mgl(1-\cos\theta)$$

Work done by torque $\delta W = \tau\,\delta\phi$

<u>Definition</u>: No. of coords that can vary independently is No. of degrees of freedom. Here: 2 deg. of freedom. (θ, ϕ)

<u>Lagrange eqns</u>

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) = \frac{\partial L}{\partial \phi} + \tau$$

$$\Rightarrow m l^2 \ddot{\theta} = m l^2 \dot{\phi}^2 \sin\theta\cos\theta - mgl\sin\theta$$

$$\frac{d}{dt}\left(m l^2 \dot{\phi} \sin^2\theta\right) = \tau$$

Now suppose τ is adjusted to constrain the system to rotate with const. ang. speed ω about vertical

constraint : $\dot{\phi} = \omega$

Now only <u>1 degree</u> of freedom : θ

sub. constraint into L :

$$L = \frac{1}{2} m l^2 \left(\dot{\theta}^2 + \omega^2 \sin^2\theta\right) - mgl(1-\cos\theta)$$

$$= T' - V'$$

redefine :

$$\underbrace{T' = \frac{1}{2} m l^2 \dot{\theta}^2}_{\text{quadratic in } \dot{\theta}} \qquad V' = \underbrace{mgl(1-\cos\theta) - \tfrac{1}{2} m l^2 \omega^2 \sin^2\theta}_{\text{independent of } \dot{\theta}}$$

centrifugal term

Same as going into rotating frame

In a forced system, constraining forces can do work

so $T + V \neq \text{const}$.

But there still is a conservation law:

multiply $ml^2\ddot{\theta} = ml^2\dot{\phi}^2 \sin\theta\cos\theta - mgl\sin\theta$ by $\dot{\theta}$

integrate over time

$\Rightarrow T' + V' = E' = \text{const}$.

This is <u>not</u> total energy $T+V$ because centrifugal term appears with opposite sign.

$T' + V' = T + V - ml^2\omega^2\sin^2\theta$.

<u>Details</u>:

$$\int ml^2\ddot{\theta} \times \dot{\theta}\, dt = \frac{1}{2} ml^2 \int \frac{d}{dt}(\dot{\theta}^2)\, dt = \frac{1}{2} ml^2\dot{\theta}^2$$

+ const of integration

$$\int \{ ml^2\omega^2 \underbrace{\sin\theta\cos\theta}_{\frac{1}{2}\frac{d}{d\theta}\sin^2\theta} - mgl\sin\theta \} \frac{d\theta}{dt}\, dt$$

$$= \frac{1}{2} ml^2\omega^2 \sin^2\theta + mgl\cos\theta + \text{const}$$

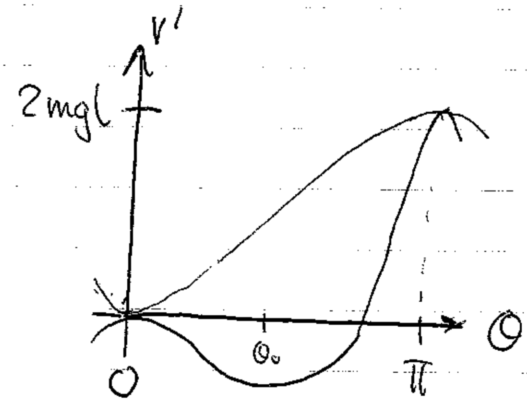
<u>So</u>

$$\underbrace{\frac{1}{2} ml^2\dot{\theta}^2}_{T'} \underbrace{- \frac{1}{2} ml^2\omega^2\sin^2\theta - mgl\cos\theta}_{+V'} = \text{Const}$$

$= \text{Const}$

cf $T + V = \frac{1}{2} ml^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta) + mgl(1 - \cos\theta)$

Effective potential V'



upper curve $\omega^2 < \frac{g}{l}$

still
V' has min at $\theta = 0$

motion qualitatively same as ordinary pendulum but period is longer

lower curve $\omega^2 > \frac{g}{l}$

V' has minimum at $\theta = \theta_0 = \arccos\left(\frac{g}{l\omega^2}\right)$

At θ_0 gravity and centrifugal force are in equilibrium

3 types of motion :

$E' < 0$ pendulum oscillates about θ_0

$0 < E' < 2mgl$ swings backwards and forwards as before though $\theta = 0$ is no longer position of maximum velocity.

$E' > 2mgl$ "windmill" i.e. continuous circular motion

We call the constraint $\dot{\phi} = \omega$ a holonomic constraint

Definition : A system is holonomic if it is possible to solve the constraint eqns, and so eliminate some of the co-ordinates, leaving a set of general coordinates equal to the number of degrees of freedom.

Here we introduced a constraint, $\dot{\phi} = \omega$, and the problem was reduced to 1 degree of freedom and one generalized coordinate, θ.

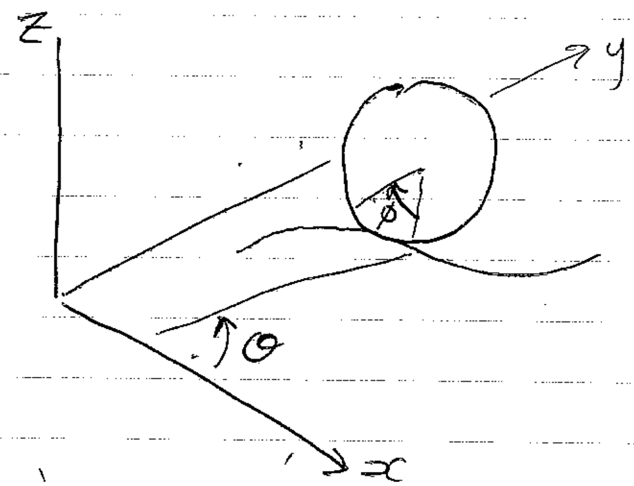
Counter ~~ex~~ example (non-holonomic)

Disk rolling on horizontal plane (xy)

Coords : (x, y, ϕ, θ)

ϕ: angle of rotation about axis of disk

θ: angle between disk and, e.g. x-axis



Rolling constraint : $v = a\dot{\phi}$ a = radius

$$V_x = \dot{x} = v\sin\theta$$

$$V_y = \dot{y} = -v\cos\theta$$

Combine to give two differential equations of constraint :

$$dx - a\sin\theta\, d\phi = 0$$

$$dy + a\cos\theta\, d\phi = 0$$

Cannot integrate either without first solving the problem.
(cannot find integrating factor $f(x, y, \theta, \phi)$ that turns either eqns into perfect differentials). Thus : cannot solve constraint equation and eliminate some coords.

Physical reason : no direct functional relation between ϕ and (x, y, θ)
because can ~~also~~ always roll disk in circle of arbitrary radius.
~~x, y, θ return to original values but ϕ depends on radius.~~
Thus need to know precise path - state of system not just function of "position" (x, y, θ)
- still need ϕ.

More general method for constraints (holonomic and nonholonomic)

Method of Lagrange multipliers: λ

$$L = \tfrac{1}{2} m l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) - mgl(1-\cos\theta) - \lambda(\dot{\phi} - \omega)$$

EL for θ unchanged

EL for ϕ

$$\frac{d}{dt}\left(ml^2 \dot{\phi} \sin^2\theta - \lambda\right) = 0$$

EL for λ

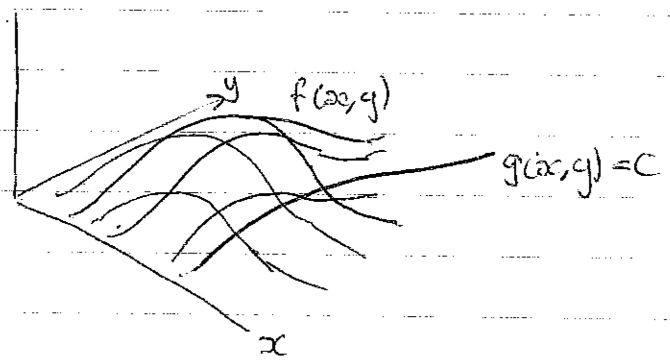
$$0 = \dot{\phi} - \omega$$

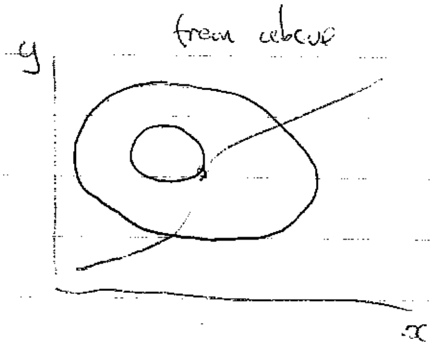
(reproduces constraint)

$$\dot{\lambda} = \tau \quad \text{torque}$$

This is an example of finding a minimum or maximum of a function subject to constraints. Example, minimize $f(x,y)$ subject to $g(x,y) = C$

Consider $A(x,y,\lambda) = f(x,y) + \lambda(g(x,y) - c)$





General idea

Walk along $g = c$.

In general $g=c$ intersects contours of f, ie f varies along $g=c$.

However when contour $g=c$ meets a contour of f tangentially, then we don't ~~edd~~ increase or decrease f either.

Contours of f and g touch when their tangent vectors are parallel, and so their gradients are also parallel.

Need: $\vec{\nabla}_{x,y} f = -\lambda \vec{\nabla}_{x,y} g$ (magnitudes not generally equal)

$$g = c$$

So solve $\vec{\nabla}_{x,y,\lambda} A(x, y, \lambda) = 0$