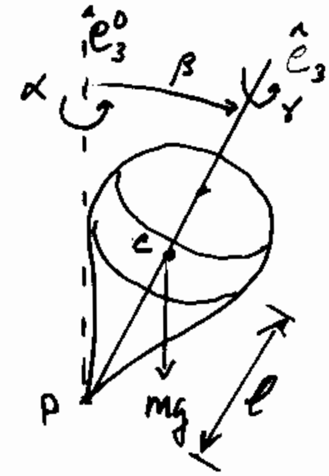
Symmetric Top, one point fixed P
in gravitational field
P = origin



$$L = \frac{1}{2} I_1 \left(\dot{\alpha}^2 \sin^2\beta + \dot{\beta}^2 \right) + \frac{1}{2} I_3 \left(\dot{\alpha} \cos\beta + \dot{\gamma} \right)^2 - mg\ell\cos\beta$$

$\dot{\alpha}, \dot{\gamma}$ coupled ; gravity acts on β : $I_1 \ddot{\beta} \neq 0$

Constants of motion:

① $$\frac{\partial L}{\partial \dot{\alpha}} \equiv P_\alpha = I_1 \dot{\alpha} \sin^2\beta + I_3 \cos\beta \left(\dot{\alpha} \cos\beta + \dot{\gamma} \right)$$

② $$\frac{\partial L}{\partial \dot{\gamma}} \equiv P_\gamma = \underline{\underline{I_3}} \left(\dot{\alpha} \cos\beta + \underline{\underline{\dot{\gamma}}} \right) = I_3 \omega_3$$

"spin" around symmetry axis

① $- \cos\beta$ ② $\Rightarrow$ $$I_1 \dot{\alpha} \sin^2\beta = P_\alpha - P_\gamma \cos\beta$$

sub. into ② $\Rightarrow$ $$I_3 \dot{\gamma} = P_\gamma - I_3 \cos\beta \left[\frac{P_\alpha - P_\gamma \cos\beta}{I_1 \sin^2\beta} \right]$$

$$I_3 \dot{\gamma} = P_\gamma \left[1 - \frac{I_3 \cos^2\beta}{I_1 \sin^2\beta} \right] - \frac{I_3}{I_1} \frac{\cos\beta}{\sin^2\beta} P_\alpha$$

$$\boxed{\begin{aligned} \dot{\gamma} &= P_\gamma \left[\frac{1}{I_3} + \frac{1}{I_1} \cot^2\beta \right] - P_\alpha \frac{\cos\beta}{I_1 \sin^2\beta} \\ \dot{\alpha} &= \frac{P_\alpha - P_\gamma \cos\beta}{I_1 \sin^2\beta} \end{aligned}}$$

$= \dot{\gamma}(\beta)$

$= \dot{\alpha}(\beta)$

P_α, P_γ I_1 I_3 are all constants. β varies

Equation of Motion for β : inclination of the symmetry axis

$$I_1 \ddot{\beta} = \frac{\partial L}{\partial \beta} = I_1 \dot{\alpha}^2 \sin\beta \cos\beta - \underline{I_3} \dot{\alpha} \sin\beta \, \underline{(\dot{\alpha}\cos\beta + \dot{\gamma})} + mgl \sin\beta \neq 0$$

But we know $\dot{\alpha}(\beta)$, $\dot{\gamma}(\beta)$ so

$$I_1 \ddot{\beta} = I_1 \sin\beta\cos\beta \left(\frac{p_\alpha - \tilde{p}_\gamma \cos\beta}{I_1 \sin^2\beta} \right)^2 - \underline{p_\gamma} \sin\beta \left(\frac{p_\alpha - \tilde{p}_\gamma \cos\beta}{I_1 \sin^2\beta} \right) + mgl \sin\beta$$

(1) $$= \frac{\cos\beta}{I_1 \sin^3\beta} \big(\underbrace{p_\alpha^2 - 2p_\alpha p_\gamma \cos\beta + p_\gamma^2}_{\text{lies between } (p_\alpha + p_\gamma)^2 \text{ and } (p_\alpha - p_\gamma)^2} \big) - \frac{p_\alpha p_\gamma}{I_1 \sin\beta} + mgl\sin\beta$$

diverges for $\beta \to 0$ or $\beta \to \pi$

This looks pretty nasty, so let's use $H = T + V = E = \text{const.}$

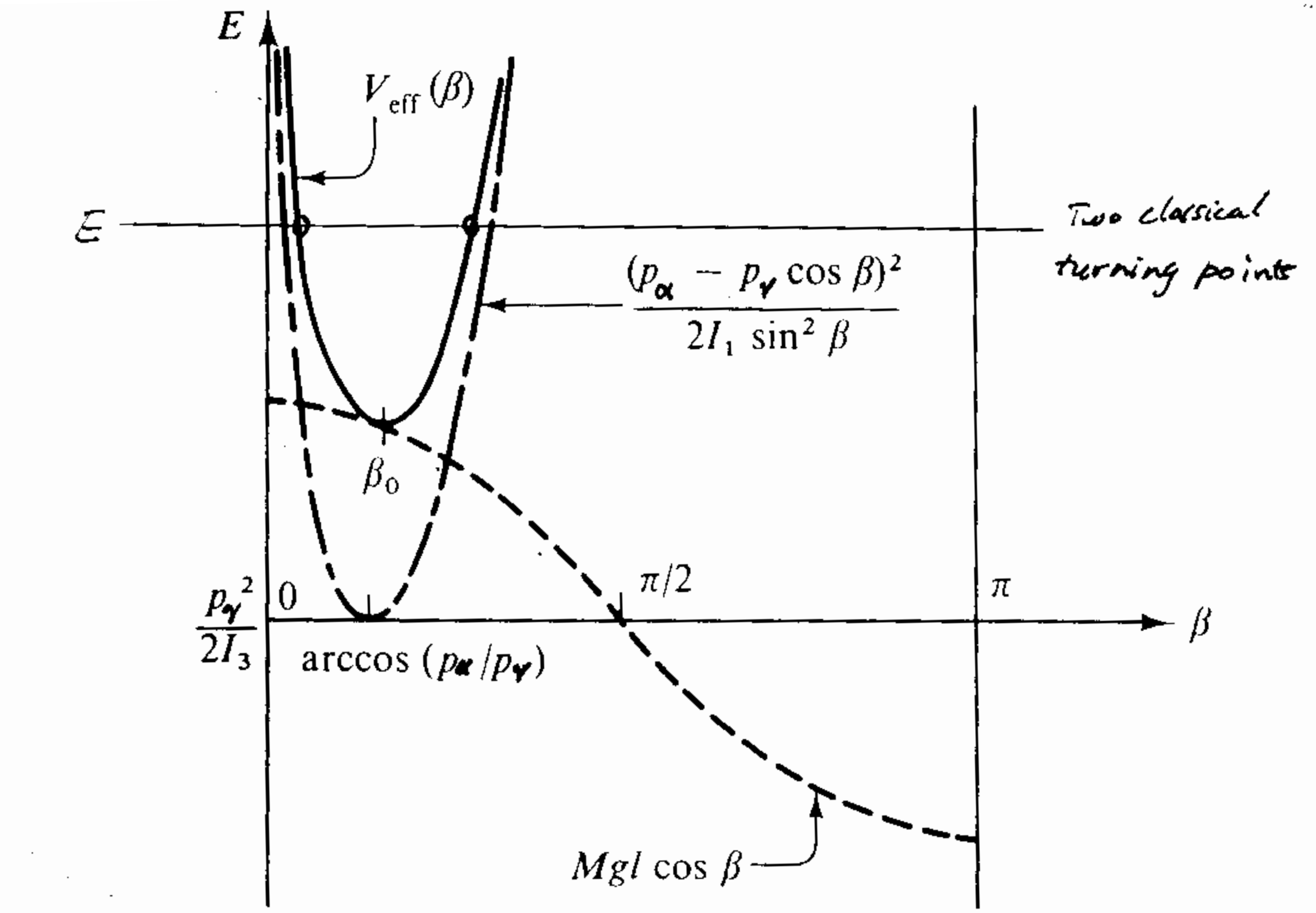
T is quadratic in $\dot{\alpha}\ \dot{\beta}\ \dot{\gamma}$ and $V = V(\beta)$, no t-dependence
$\Rightarrow$ Energy is conserved

Write $$E = \frac{1}{2} I_1 \dot{\beta}^2 + V_{eff}(\beta) = \frac{1}{2} p_\beta^2 / I_1 + V_{eff}(\beta)$$

$$V_{eff}(\beta) \equiv \frac{(p_\alpha - p_\gamma \cos\beta)^2}{2 I_1 \sin^2\beta} + \overbrace{\frac{p_\gamma^2}{2 I_3}}^{\text{constant} \equiv V_0} + mgl\cos\beta$$

$$-\frac{\partial V_{eff}}{\partial \beta} = \frac{\cos\beta}{I_1 \sin^3\beta} (p_\alpha - p_\gamma \cos\beta)^2 - \frac{\sin\beta \, p_\gamma}{I_1 \sin^2\beta} (p_\alpha - p_\gamma \cos\beta) + mgl\sin\beta$$

$$= I_1 \ddot{\beta}$$ which agrees with (1)



$$V_{\text{eff}}(\beta) = \frac{(p_\alpha - p_\gamma\cos\beta)^2}{2I_1\sin^2\beta} + Mgl\cos\beta + \left(V_0 = \frac{p_\gamma^2}{2I_3}\right)$$

$\dfrac{\partial V_{\text{eff}}}{\partial\beta} = 0$ at $\beta = \beta_0$ $\qquad 0 < \beta < \pi$

Figure 31.2 Effective potential $V_{\text{eff}}(\beta)$ in Eq. (31.7) for system illustrated in Fig. 31.1.

$\beta = 0$ is vertical top

$= \pi$ is inverted.

For a fast spinning top, p_α large $(p_\alpha \pm p_\gamma)^2$ large also

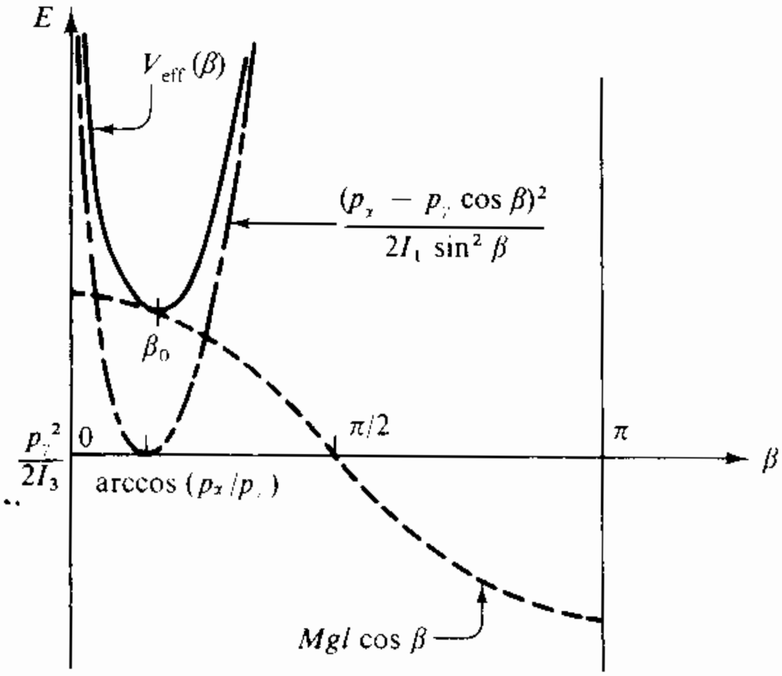
Except for angles close to $(p_\alpha - p_\gamma\cos\beta) = 0$, $V_{\text{eff}} \gg V_0$

Just as for the orbits in $1/r$ potential, here β is limited to a narrow range between the two turning points

When $p_\alpha, p_\gamma > 0$ $\quad \beta_0 < 90°$ so top sits "upright".

Effective Potential

Consider small oscillations around the minimum at $\beta = \beta_0$: $\beta = \beta_0 + \eta$



$$V \simeq V(\beta_0) + \eta\left(\frac{\partial V}{\partial \beta}\right)_0 + \frac{\eta^2}{2}\left(\frac{\partial^2 V}{\partial \beta^2}\right)_0 + \cdots$$

zero

$$-\frac{\partial V}{\partial \beta} = \frac{\cos\beta}{I_1 \sin^3\beta}\left(\underbrace{p_\alpha^2 - 2p_\alpha p_\gamma \cos\beta + p_\gamma^2}_{Z_{\alpha\beta}}\right) - \frac{p_\alpha p_\gamma}{I_1 \sin\beta} + mgl\sin\beta \longrightarrow 0$$

$$-\frac{\partial^2 V}{\partial \beta^2} = \frac{-3\cos^2\beta}{I_1 \sin^4\beta} Z_{\alpha\beta} - \frac{1}{I_1 \sin^2\beta} Z_{\alpha\beta} + \frac{2p_\alpha p_\gamma \cos\beta}{I_1 \sin^2\beta} + \frac{p_\alpha p_\gamma \cos\beta}{I_1 \sin^2\beta} + Mgl\cos\beta$$

$$-\frac{\partial^2 V}{\partial \beta^2}\bigg|_{\beta_0} = -\frac{3\cos\beta_0}{\sin\beta_0}\left[\frac{p_\alpha p_\gamma}{I_1 \sin\beta_0} - Mgl\sin\beta_0\right] + 3\frac{p_\alpha p_\gamma \cos\beta_0}{I_1 \sin^2\beta_0} - \frac{\sin\beta_0}{\cos\beta_0}\left[\qquad\qquad\right] + Mgl\cos\beta_0$$

$$= \frac{p_\alpha p_\gamma}{I_1 \sin^2\beta_0}\left[-3\cos\beta_0 - \frac{\sin^2\beta_0}{\cos\beta_0} + 3\cos\beta_0\right] + \frac{Mgl}{\cos\beta_0}\left(3\cos^2\beta_0 + \sin^2\beta_0 + \cos^2\beta_0\right)$$

$$= \frac{-p_\alpha p_\gamma}{I_1 \cos\beta_0} + \frac{Mgl}{\cos\beta_0}\left[4 - 3\sin^2\beta_0\right]$$

$$I_1 \ddot{\eta} = -\eta \left[\frac{\partial^2 V}{\partial \beta^2}\right]_{\beta_0} = -I_1 \Omega^2 \eta$$

$$\Omega^2 = \frac{1}{I_1}\left(\frac{\partial^2 V}{\partial \beta^2}\right)_{\beta_0} = \frac{p_\alpha p_\gamma - I_1 mgl\,[4 - 3\sin^2\beta_0]}{I_1^2 \cos\beta_0}$$

System will undergo small oscillations if $\Omega^2 > 0$

Need a large value of $p_\alpha p_\gamma$: $p_\gamma = I_3 \omega_3$

ω_3 = angular velocity component on symmetry axis

"Sleeping Top" – steady motion with symmetry axis vertical : $\beta_0 \to 0$ $[\hat{e}_\alpha = \hat{e}_\gamma]$ $p_\alpha = p_\gamma$

Occurs with $\Omega^2 = \dfrac{p_\gamma^2 - I_1 mgl\,[4]}{I_1^2} > 0$

In practice, friction slows down the top; $\omega_3 \downarrow$ to point where the top begins to lean over and wobble: Symmetry axis precesses around the vertical.

$$\omega_3^2 > 4 \frac{I_1}{I_3^2} mgl \qquad I_3 < I_1 \text{ for typical top.}$$

For smaller ω_3, symmetry axis must precess.

Then $\beta(t) \simeq \beta_0 + \eta_0 \cos(\Omega t + \varphi_0)$ $\quad$ η_0, φ_0 depend on initial cond^s.

Precession and Nutation of the Symmetrical Top

Recall that $\dot\alpha = \dfrac{p_\alpha - p_\gamma \cos\beta}{I_1 \sin^2\beta}$, $\dot\gamma = p_\gamma - \dfrac{I_3}{I_1}\cos\beta\,\dot\alpha$

When $\beta = \beta_0 + \eta(t)$ $\cos\beta \simeq \cos\beta_0 - \sin\beta_0\,\eta(t)\cdots$

$$= \cos\beta_0\,(1 - \tan\beta_0\,\eta(t)\cdots)$$

$$\sin\beta \simeq \sin\beta_0 + \cos\beta_0\,\eta(t)\cdots$$

$$= \sin\beta_0\,(1 + \cot\beta_0\,\eta(t)$$

$$\dot\alpha \simeq \frac{p_\alpha - p_\gamma\cos\beta_0\,(1 - \tan\beta_0\,\eta\cdots)}{I_1\sin^2\beta_0\,(1+\cot\beta_0\,\eta\cdots)^2} \simeq \dot\alpha_0 + \dot\alpha_1\,\eta(t) + \cdots$$

$$\dot\gamma \simeq p_\gamma - \frac{I_3}{I_1}\,\dot\alpha_0 \,{}^{*}(1+\dot\alpha_1\eta/\dot\alpha_0)\cos\beta_0\,(1 - \tan\beta_0\,\eta(t)\cdots) \simeq \dot\gamma_0 + \dot\gamma_1\,\eta(t)\cdots$$

$\dot\alpha(t)$ is rate of precession of symmetry axis around vertical

mean value $\dot\alpha_0$, oscillating component $\dot\alpha_1\,\eta(t)$

$\dot\gamma(t)$ is rate of spin around the symmetry axis,

mean value $\dot\gamma_0$, oscillating component $\dot\gamma_1\,\eta(t)$

can only be seen using a strobe light.

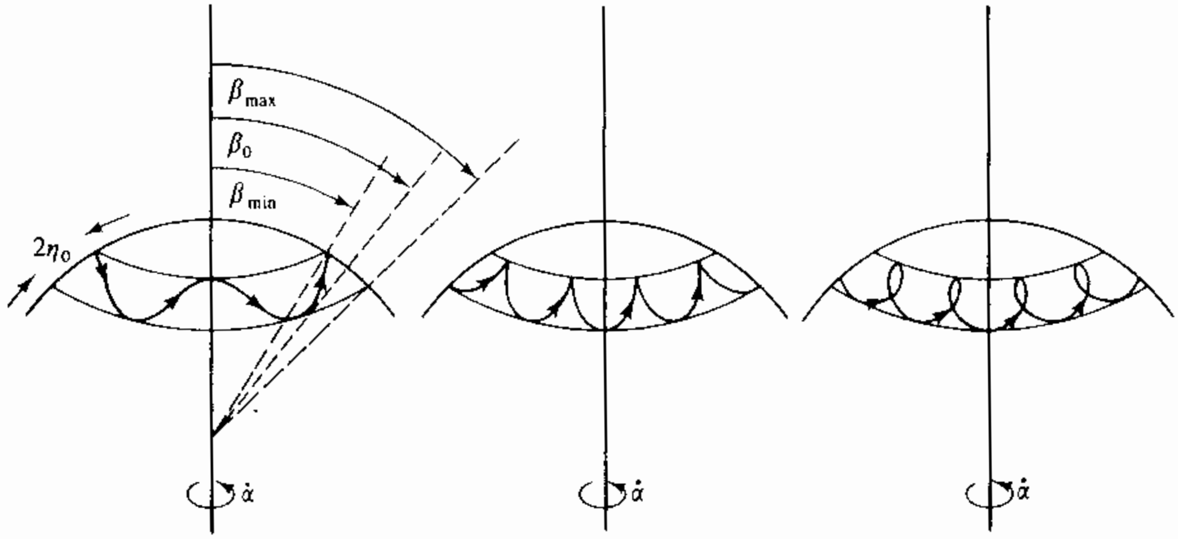
$\eta(t)$ is variation in tilt of the symmetry axis, called

nutation or "nodding".

Relative size of $\dot\alpha_0$, $\dot\alpha_1$ leads to many possible

forms for the motion of the symmetry axis

see Fig. 31.3



(a) (b) (c)

Figure 31.3 Nutation of a symmetric top: (a) $(\dot\alpha)_0 > \eta_0(\dot\alpha)_1$; (b) $(\dot\alpha)_0 = \eta_0(\dot\alpha)_1$; (c) $(\dot\alpha)_0 < \eta_0(\dot\alpha)_1$.

If $\dot\alpha_1 > 0$ then largest $\dot\alpha(t)$ occurs at largest $\beta(t)$
smallest " " " smallest " .

initial values of η_0, $\dot\alpha_0$, $\dot\gamma_0$ affect path taken.

Fig (a) $\dot\alpha(t)$ always positive (b) $\dot\alpha(t)$ vanishes at minimum β
(c) retrograde motion of symmetry axis at minimum β.

$\Omega \Rightarrow$ frequency of $\eta(t)$, $\dot\alpha_0 \Rightarrow$ average rate of precession.

unless $\Omega/\dot\alpha_0$ = rational fraction, the path taken by the symmetry axis will never repeat exactly.