Motion of Symmetric Rigid Body in Inertial Frame

$$\underset{\sim}{L} = I_1 \omega_1 \hat{e}_1 + I_2 \omega_2 \hat{e}_2 + I_3 \omega_3 \hat{e}_3$$

$$\neq p_\alpha \hat{e}_\alpha + p_\beta \hat{e}_\beta + p_\gamma \hat{e}_\gamma$$

BUT $p_\alpha = \underset{\sim}{L} \cdot \hat{e}_\alpha$

$p_\beta = \underset{\sim}{L} \cdot \hat{e}_\beta$

$p_\gamma = \underset{\sim}{L} \cdot \hat{e}_\gamma$

When no torques act, $\underset{\sim}{L}$ is fixed. To simplify discussion

CHOOSE $\underset{\sim}{L}$ to lie along $\hat{e}_3^{\,0} = \hat{e}_\alpha$

$$\underset{\sim}{L} = |L| \hat{e}_\alpha \quad \Rightarrow \quad |L| = p_\alpha = I_1 \sin^2\beta\, \dot{\alpha} + p_\gamma \cos\beta$$

$$p_\gamma = \underset{\sim}{L} \cdot \hat{e}_3 = I_3 \omega_3 = I_3 (\dot{\alpha} \cos\beta + \dot{\gamma})$$

But: $\underset{\sim}{L}$ $\parallel^{el}$ $\hat{e}_\alpha$ and projection $\hat{e}_\alpha \cdot \hat{e}_\gamma = \cos\beta$

$$\therefore\ p_\gamma = |L| \cos\beta = p_\alpha \cos\beta$$

p_γ is constant, so is p_α $\Rightarrow$ angle β fixed

(Look at Poinsot construction when $I_1 = I_2$,
so polhode is a circle $\perp_r$ to symmetry axis)

β fixed $\Rightarrow$ $\dot{\alpha}$ = constant $\quad \alpha = \alpha_0 + \dot{\alpha} t$ increases linearly.

Then p_γ = const $\Rightarrow$ $\dot{\gamma}$ = constant $\quad \gamma = \gamma_0 + \dot{\gamma} t$ " "

β fixed $\Rightarrow$ $p_\beta = I_1 \dot{\beta} = 0$

$$\frac{d}{dt}(p_\beta) = \frac{\partial T}{\partial \beta} = \dot{\alpha} \sin\beta \left[I_1 \dot{\alpha} \cos\beta - I_3 \omega_3 \right] = 0$$

$$\therefore\ \boxed{\begin{aligned} \dot{\alpha} \cos\beta &= \frac{I_3 \omega_3}{I_1} \\ \dot{\gamma} &= \omega_3 \left(\frac{I_1 - I_3}{I_1} \right) \end{aligned}}$$

$\dot{\alpha}$ "spin" along $\hat{e}_3$ axis

$\dot{\alpha} \cos\beta + \dot{\gamma} = \omega_3$

$$\underset{\sim}{\omega} = \dot{\alpha}\,\hat{e}_\alpha + \dot{\gamma}\,\hat{e}_\gamma = \frac{I_3}{I_1}\omega_3\,\underset{\sim}{\hat{e}_3^{\,0}} + \left(\frac{I_1 - I_3}{I_1}\right)\omega_3\,\underset{\sim}{\hat{e}_3}$$

$(\dot{\beta} = 0)$

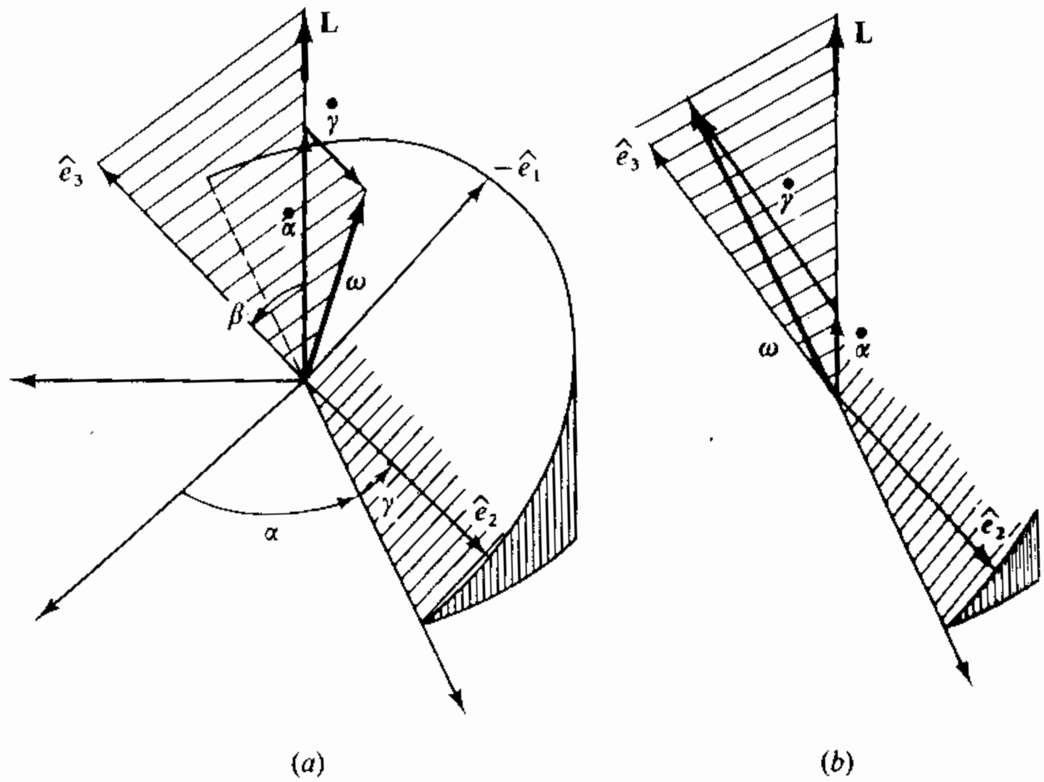

Figure 30.1 Torque-free motion of a symmetric top seen from an inertial frame: (*a*) $I_3 > I_1$; $\dot{\gamma} < 0$; (*b*) $I_3 < I_1$; $\dot{\gamma} > 0$.

$\underset{\sim}{\omega}$ lies in the plane defined by the symmetry axis $\hat{e}_3$ and by $\underset{\sim}{L}$ $(\hat{e}_3^{\,0})$

Oblate body like earth: $I_3 > I_1$ $\quad \dot{\gamma} < 0 \quad$ $\hat{e}_3$ precesses around $\underset{\sim}{L}$ in positive sense, at fixed polar angle β.

Body spins around $\hat{e}_3^{\,0}$ at $\omega_\alpha = \dot{\alpha}$ uniformly.

$\dot{\gamma} = \dfrac{I_1 - I_3}{I_1}\omega_3 = -\Omega$ is negative $\simeq -\dfrac{\omega_3}{305}$ for earth

Inertial observer sees slow 'retrograde' rotation around $\hat{e}_3$

$|\omega_\perp| = \dot{\alpha}\sin\beta \qquad \left|\dfrac{\omega_\perp}{\omega_3}\right| = \dfrac{I_3}{I_1}\tan\beta = \tan\lambda \qquad$ $\lambda > \beta$ is angle between $\underset{\sim}{\omega}$ and $\hat{e}_3$ both SMALL angles

Symmetry axis precesses around $\underset{\sim}{L}$ at

$\dot{\alpha}\cos\beta \sim \dfrac{306}{305}\,\omega_3 \simeq$ once per day.

Elongated or PROLATE body : $I_3 < I_1 = I_2$

$$\dot{\alpha} \cos\beta = \frac{I_3}{I_1}\omega_3 < \omega_3$$

$$I_3 = \int \rho(r)\, d^3r\, (x^2 + y^2) = \int \rho(r)\, d^3r\, r^2 (1 - \cos^2\theta)$$

$$2 I_1 = \int \rho(r)\, d^3r \left[(y^2 + z^2) + (x^2 + z^2) \right] = \int \rho(r)\, r^2 d^3r\, (\sin^2\theta + 2\cos^2\theta)$$

$$\frac{I_3}{I_1} \simeq \frac{2\int \cdots \sin^2\theta}{\int \cdots (1 + \cos^2\theta)}$$

$\leftarrow$ emphasises $\theta \sim 90° \sim xy$ plane

$\leftarrow$ " $\theta \sim 0, \pi$ ~ poles

$$0 < \frac{I_3}{I_1} < 2$$

can be very small but never too large.

pencil CD

Fig 30.1 (b) is for $I_3 \ll I_1$ $\quad \dot{\alpha}\cos\beta \ll \omega_3$

$$\dot{\gamma} \approx +\omega_3 \simeq \omega$$

Symmetry axis rotates slowly, positive sense, around $\underset{\sim}{L}$

Body rotates rapidly around symmetry axis $\hat{e}_3$ at $\dot{\gamma} \approx \omega$

$\omega_\perp = \dot{\alpha} \sin\beta \ll \omega_3$.