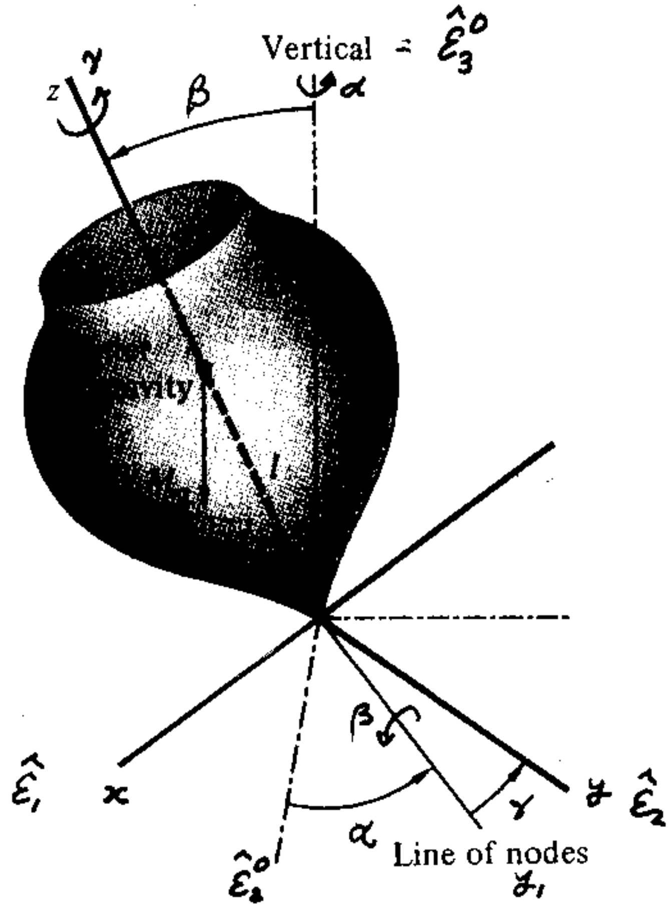
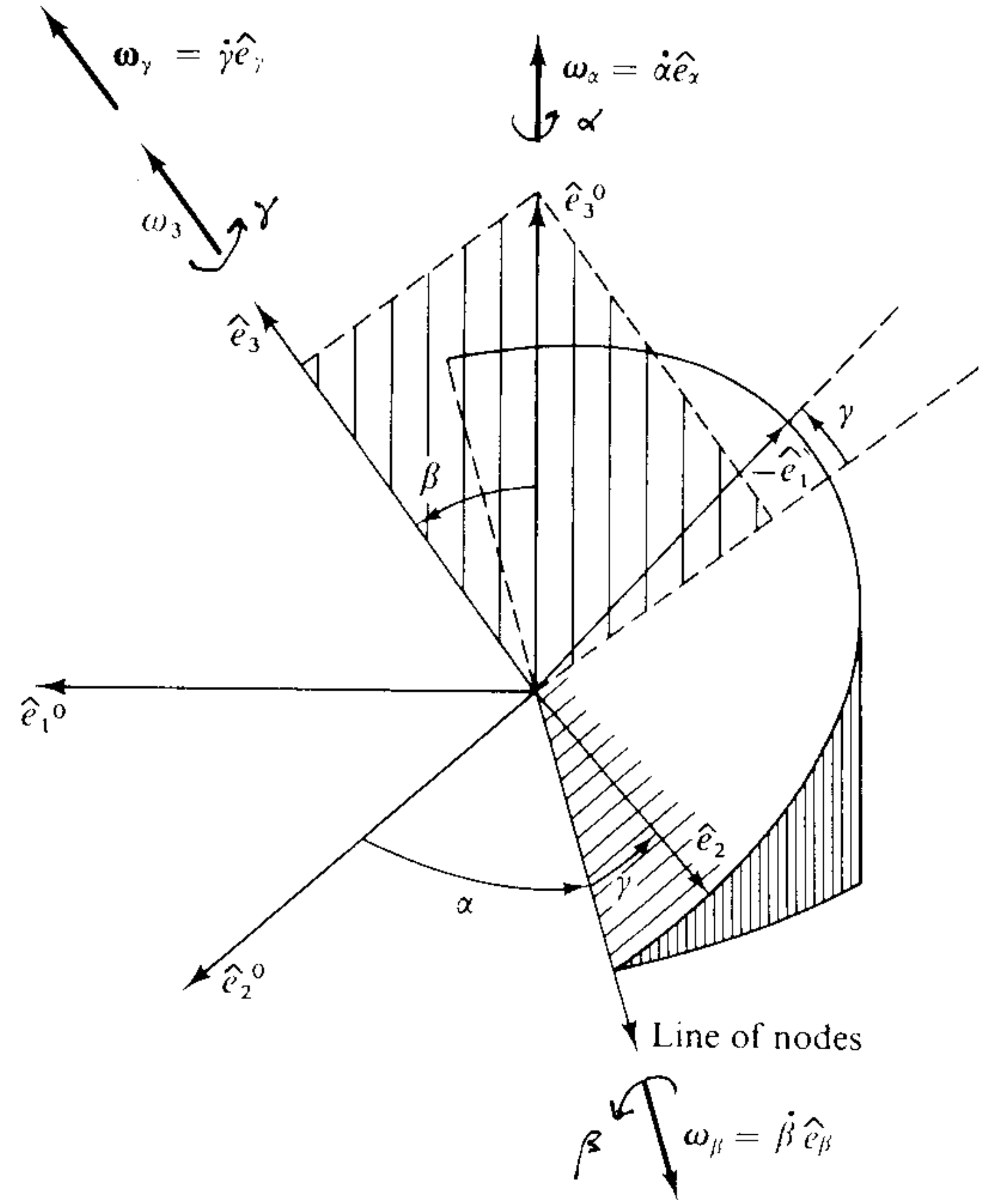


FIGURE 5.7 Euler's angles specifying the orientation of a symmetrical top.

We follow the convention of Quantum Mechanics in our choice of Euler Angles

1. rotate around $\hat{\varepsilon}_3^0$ by angle α
2. " " intermediate y axis by angle β

   $\Rightarrow$ Symmetry axis of top has polar angles $\theta = \beta$, $\phi = \alpha$
3. rotate around $\hat{\varepsilon}_3$ by angle γ to specify "final" position of the top's body fixed axes $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$.



$0 < \alpha < 2\pi$

$0 < \beta < \pi$

$0 < \gamma < 2\pi$

Figure 29.1 Definition of Euler's angles α, β, γ.

$$\underset{\sim}{\omega} = \omega_\alpha \hat{\underset{\sim}{e}}_\alpha + \omega_\beta \hat{\underset{\sim}{e}}_\beta + \omega_\gamma \hat{\underset{\sim}{e}}_\gamma$$ in non-orthogonal basis

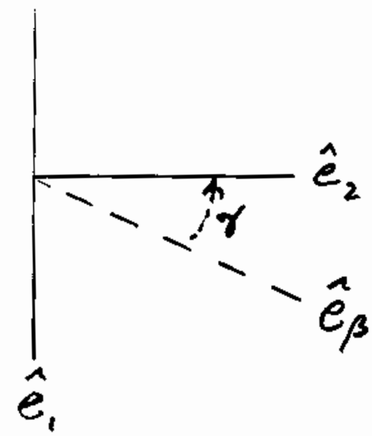
$$= \omega_1 \hat{\underset{\sim}{e}}_1 + \omega_2 \hat{\underset{\sim}{e}}_2 + \omega_3 \hat{\underset{\sim}{e}}_3$$ body-fixed axes

We want to express $\underset{\sim}{\omega}$ in the body fixed (final) axes $\hat{e}_1$ $\hat{e}_2$ $\hat{e}_3$: $\omega = \omega_1 \hat{e}_1 + \omega_2 \hat{e}_2 + \omega_3 \hat{e}_3$

$\dot{\gamma}$ 1. $\boxed{\hat{e}_3 = \hat{e}_\gamma}$ $\Rightarrow$ $\omega_\gamma \hat{e}_\gamma = \omega_\gamma \hat{e}_3$. Done

2. $\hat{e}_\beta$ lies in the $\hat{e}_1$, $\hat{e}_2$ plane

rotation by γ brings it to $\hat{e}_2$.



$\dot{\beta}$ $\boxed{\hat{e}_\beta = \hat{e}_1 \sin\gamma + \hat{e}_2 \cos\gamma}$

3. $\hat{e}_\alpha = \hat{e}_3^0$ has polar angle $+\beta$

azimuthal angle $(\pi - \gamma)$ measured from $\hat{e}_1$

(see figure 29.1)

In general $\underset{\sim}{r} = r[\sin\theta \cos\phi, \ \sin\theta \sin\phi, \ \cos\theta]$

$\dot{\alpha}$ $\boxed{\therefore \hat{e}_\alpha = -\sin\beta \cos\gamma \, \hat{e}_1 + \sin\beta \sin\gamma \, \hat{e}_2 + \cos\beta \, \hat{e}_3}$

$$\omega_1 = -\dot{\alpha} \sin\beta \cos\gamma + \dot{\beta} \sin\gamma$$

$$\omega_2 = \dot{\alpha} \sin\beta \sin\gamma + \dot{\beta} \cos\gamma$$

$$\omega_3 = \dot{\alpha} \cos\beta + \dot{\gamma}$$

$$\omega_1^2 + \omega_2^2 = \dot{\beta}^2 + \dot{\alpha}^2 \sin^2\beta$$

$$2T = I_1 \omega_1^2 + I_2 \omega_2^2 + I_3 \omega_3^2$$

$$\rightarrow I_1 (\dot{\beta}^2 + \dot{\alpha}^2 \sin^2\beta) + I_3 (\dot{\alpha}\cos\beta + \dot{\gamma})^2$$

for a symmetric top with $I_1 = I_2$

Motion of Symmetric top, Body-fixed frame:

$$L = T = \frac{1}{2} I_1 \left(\dot{\beta}^2 + \sin^2\beta\, \dot{\alpha}^2 \right) + \frac{1}{2} I_3 \left(\dot{\alpha} \cos\beta + \dot{\gamma} \right)^2$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\alpha}} \right) - \frac{\partial L}{\partial \alpha} = 0$$

α, β, γ taken as the generalised coordinates

$P_\alpha \equiv \dfrac{\partial L}{\partial \dot{\alpha}}$ is a constant of motion

$$p_\gamma = \frac{\partial L}{\partial \dot{\gamma}} \text{ also, } = I_3 (\dot{\alpha} \cos\beta + \dot{\gamma}) = I_3 \omega_3 = L_3$$

$$\begin{aligned} P_\alpha &= I_1 \sin^2\beta\, \dot{\alpha} + I_3 \cos\beta (\dot{\alpha} \cos\beta + \dot{\gamma}) \\ &= I_1 \sin^2\beta\, \dot{\alpha} + \cos\beta (p_\gamma) = \text{constant} \\ &= \underset{\sim}{L} \cdot \underset{\sim}{\hat{e}_\alpha} = \text{projection of } \underset{\sim}{L} \text{ on } \hat{e}_3^0 \end{aligned}$$

Projection of $\underset{\sim}{L}$ on symmetry axis AND on the inertial OZ are BOTH constants of motion (for symmetric top).

proof:

$$\hat{e}_\alpha = \left(-\sin\beta \cos\gamma ,\ \sin\beta \sin\gamma ,\ \cos\beta \right)$$

$$\underset{\sim}{L} = \Big(I_1 (-\dot{\alpha} \sin\beta \cos\gamma + \dot{\beta} \sin\gamma),\ I_1 (\dot{\alpha} \sin\beta \sin\gamma + \dot{\beta} \cos\gamma),\ I_3 (\dot{\alpha} \cos\beta + \dot{\gamma}) \Big)$$

$$I_1 \left[\dot{\alpha} \sin^2\beta (\cos^2\gamma + \sin^2\gamma) + 0 + 0 \right] + \cos\beta \cdot p_\gamma$$

QED

$P_\beta = I_1 \dot{\beta} = \underset{\sim}{L} \cdot \hat{e}_\beta$ can be seen similarly.

NOT A CONSTANT OF MOTION

A top in gravitation field also has a force acting in β direction.

$\underset{\sim}{\omega}$, $\underset{\sim}{L}$ and the symmetry axis are co-planar for symmetric case

For completeness

$$\underset{\sim}{L} = \Big(I_1(-\dot\alpha \sin\beta \cos\gamma + \dot\beta \sin\gamma),\ I_1(\dot\alpha \sin\beta \sin\gamma + \dot\beta \cos\gamma),\ I_3 \omega_3 \Big)$$

$$\hat{e}_\alpha = (-\sin\beta \cos\gamma,\ \sin\beta \sin\gamma,\ \cos\beta)$$

$$\underset{\sim}{L} \cdot \underset{\sim}{\hat{e}_\alpha} = I_1 \dot\alpha \sin^2\beta (\cos^2\gamma + \sin^2\gamma) + \dot\beta(0) + \underbrace{I_3 \omega_3}_{p_\gamma} \cos\beta$$

$$= p_\alpha$$

$$\hat{e}_\beta = (\sin\gamma,\ \cos\gamma,\ 0)$$

$$\underset{\sim}{L} \cdot \underset{\sim}{\hat{e}_\beta} = \dot\alpha(0) + I_1 \dot\beta (\sin^2\gamma + \cos^2\gamma) + 0 = I_1 \dot\beta$$

$$= p_\beta$$

$$\hat{e}_\gamma = (0,\ 0,\ 1)$$

$$\underset{\sim}{L} \cdot \underset{\sim}{\hat{e}_\gamma} = p_\gamma$$

$$T = \tfrac{1}{2} I_1 (\dot\beta^2 + \sin^2\beta\, \dot\alpha^2) + \tfrac{1}{2} I_3 (\dot\alpha \cos\beta + \dot\gamma)^2$$

$$p_\alpha = \frac{\partial T}{\partial \dot\alpha} = I_1 \sin^2\beta\, \dot\alpha + I_3 \cos\beta (\dot\alpha \cos\beta + \dot\gamma) = \text{const}$$

$$p_\gamma = \frac{\partial T}{\partial \dot\gamma} = I_3 (\dot\alpha \cos\beta + \dot\gamma) = \text{const}$$

$$\frac{d}{dt} p_\beta = \frac{\partial T}{\partial \beta} = I_1 \dot\alpha^2 \sin\beta \cos\beta - I_3 (\underbrace{\dot\alpha \cos\beta + \dot\gamma}_{\omega_3}) \sin\beta\, \dot\alpha$$

$$= \dot\alpha \sin\beta \left[I_1 \dot\alpha \cos\beta - I_3 \omega_3 \right]$$