The Ellipsoid of Inertia: a neat way to view the inertial properties of an irregular body

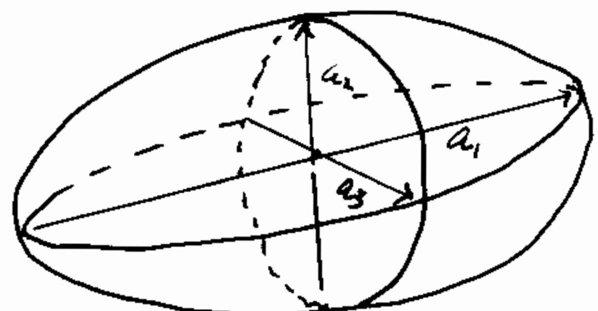
$$2T = \underset{\sim}{\omega}^T \hat{I} \underset{\sim}{\omega} \qquad\qquad \underset{\sim}{L} = I\underset{\sim}{\omega}$$

Let $\underset{\sim}{x} = \underset{\sim}{\omega}/\sqrt{2T}$

Then $\underset{\sim}{x}^T \hat{I} \underset{\sim}{x} = 1 = I_1 x_1^2 + I_2 x_2^2 + I_3 x_3^2$

Eq^n of an ellipsoid $\quad \equiv \dfrac{x_1^2}{a_1^2} + \dfrac{x_2^2}{a_2^2} + \dfrac{x_3^2}{a_3^2} = 1$

Largest semi-axis a_j
$\longleftrightarrow$ smallest I_j



A uniform solid object of this shape would have the same inertial properties as the original rigid body with the same I_j Principal Moments of Inertia

If $a_1 = a_2 \rightarrow$ ellipsoid of rotation around OZ
choice of $OX, OY \rightarrow$ arbitrary.

IF $\underset{\sim}{\omega} \rightarrow |\omega| \hat{n}$ with a unit vector $\hat{n}$

then $T = \frac{1}{2}\omega^2 \, \underset{\sim}{\hat{n}}^T \hat{I} \underset{\sim}{\hat{n}} \equiv \frac{1}{2}\omega^2 I(\hat{n})$

"The moment of inertia for rotations around direction $\hat{n}$

Useful in situation where rotation is constrained to be around a fixed direction $\hat{n}$: requires torque e.g. wheel on your car.

Euler's equations for rigid body motion:

Based on $\left(\frac{d\underset{\sim}{L}}{dt}\right)_{inertial} = \underset{\sim}{\Gamma}^{ext.}$

Useful in two situations:

1) Origin is in an inertial frame and torques computed around that origin

2) Origin is at the C.Mass of the rigid body and torques computed around the C.M. $\underset{\sim}{L} \to \underset{\sim}{L}'$ internal angular momentum.

We use body fixed axes $\hat{e}_1\ \hat{e}_2\ \hat{e}_3$ in either case

$$\left(\frac{dL}{dt}\right)_{in} = \left(\frac{dL}{dt}\right)_{body} + \underset{\sim}{\omega} \times \underset{\sim}{L} = \underset{\sim}{\Gamma}^{ext}$$

Use Principal axes of inertia $L_s = I_s \omega_s \quad s = 1, 2, 3$

$$\left(\frac{dL_s}{dt}\right)_{body} = I_s \dot{\omega}_s \qquad (\underset{\sim}{\omega} \times \underset{\sim}{L})_1 = \omega_2 L_3 - \omega_3 L_2$$

cyclically

$L_2 = I_2 \omega_2 \quad L_3 = I_3 \omega_3$

$$I_1 \dot{\omega}_1 = \omega_2 \omega_3 (I_2 - I_3) + \Gamma_1^{ex}$$

$$I_2 \dot{\omega}_2 = \omega_3 \omega_1 (I_3 - I_1) + \Gamma_2^{ex}$$

$$I_3 \dot{\omega}_3 = \omega_1 \omega_2 (I_1 - I_2) + \Gamma_3^{ex}$$

looks nice but the torques change as the body rotates — need to know sol^n to compute $\underset{\sim}{\Gamma}$.

To make solution easy, have to look at special cases with few degrees of freedom.

e.g. motion around a fixed axis

Example: Torque free motion of a symmetric top

$$I_1 = I_2 \neq I_3 \qquad I_3 \dot{\omega}_3 = 0 \qquad L_3 = \text{constant}$$

$$\omega_3 = \text{constant} \equiv \omega \cos\lambda$$

$$I_2 \dot{\omega}_1 = \omega_2\, \omega_3 (I_2 - I_3) \qquad \boxed{\dot{\omega}_1 = -\Omega\, \omega_2}$$

$$I_2 \dot{\omega}_2 = \omega_1\, \omega_3 (I_3 - I_2) \qquad \boxed{\dot{\omega}_2 = +\Omega\, \omega_1}$$

$$\Omega \equiv \omega_3 \left(\frac{I_3 - I_2}{I_1}\right) \qquad \text{let } z = \omega_1 + i\omega_2$$

$$\dot{z} = i\Omega(\omega_1 + i\omega_2) = i\Omega z \Rightarrow z \simeq e^{i\Omega t}$$

Suppose that at $t=0$ $\omega_1 = \omega \sin\lambda$ $\omega_2 = 0$ $\omega_3 = \omega\cos\lambda$

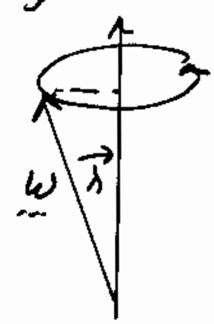
$|\omega|^2 = 1$ $t \neq 0$

$$\omega_1 = (\omega \sin\lambda) \cos \Omega t$$

$$\omega_2 = (\omega \sin\lambda) \sin \Omega t$$

$$\omega_3 = \omega \cos\lambda$$

Angular velocity vector precesses around the symmetry axis $(I_3, \vec{\xi}_3)$ at frequency Ω.



For the earth, $\dfrac{I_3 - I_1}{I_1} \sim \dfrac{1}{305}$

λ very small $\Omega \simeq \dfrac{1}{305} \cdot \omega \simeq \dfrac{2\pi}{305 \text{ days}}$

Chandler's period: mean period $\simeq$ 14 months

radius $\simeq$ 4 m. ($\lambda \sim 6 \times 10^{-7}$ radians)

But note, earth is not entirely rigid, has fluid core, maybe affected by storms

Torque-free motion of an asymmetric top:

Stability for rotation around a principal axis

Suppose $\underset{\sim}{\omega}$ is close to the 3-axis, will it remain there?

$$\omega_1 \to \eta_1(t) \quad \omega_2 \to \eta_2(t) \quad \omega_3 = \omega_0 + \eta_3(t)$$

η_i are small $\ll |\omega_0|$ Neglect terms of order η^2

$$I_1 \dot\eta_1 = (I_2 - I_3)\,\omega_0 \eta_2 \qquad I_3 \dot\omega_3 \simeq \eta_1 \eta_2 (\cdots) \to 0$$

$$I_2 \dot\eta_2 = (I_3 - I_1)\,\omega_0 \eta_1$$

$$I_1 \ddot\eta_1 = (I_2 - I_3)\omega_0 \frac{(I_3 - I_1)}{I_2}\omega_0 \eta_1 \Rightarrow \boxed{\begin{aligned}\ddot\eta_1 &= -\Omega_0^2 \eta_1 \\ \ddot\eta_2 &= -\Omega_0^2 \eta_2\end{aligned}}$$

$$\Omega_0^2 = \frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}\omega_0^2$$

Motion is stable when $\Omega_0^2 > 0 \Rightarrow$ real frequency Ω_0

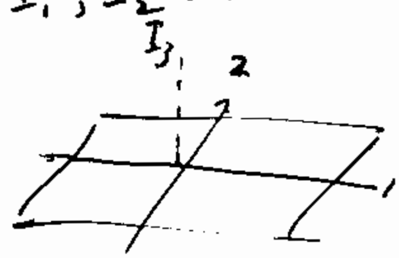
$\eta_1(t)$, $\eta_2(t)$ undergo SHM around zero.

$\Omega_0^2 > 0$ when 1) I_3 is the largest moment

2) I_3 " " smallest " .

When I_3 lies between I_1, I_2 : motion UNSTABLE

e.g. flipping cards



$I_3 = I_1 + I_2$ is largest

I_1 is smallest (stable)

I_2 is intermediate (unstable)

flipping around I_3 is cheating — most stable.

Force-free motion of a rigid body:

c.m. at rest $\quad T = \frac{1}{2}\underset{\sim}{\omega}^T \cdot \hat{I}\underset{\sim}{\omega} = \text{constant}$

$$\underset{\sim}{L} = \hat{I}\underset{\sim}{\omega} = \text{constant.}$$

$\underset{\sim}{x} \equiv \underset{\sim}{\omega}/\sqrt{2T} \qquad \underset{\sim}{x}^T \hat{I}\underset{\sim}{x} = 1$ Ellipsoid of Inertia

At a certain instant t, $\underset{\sim}{x} = \underset{\sim}{x}_p$ is a point on the ellipsoid on the instantaneous axis of rotation.

$\underset{\sim}{L} = \hat{I}\underset{\sim}{\omega} = \frac{1}{\sqrt{2T}}\hat{I}\underset{\sim}{x} \qquad \underset{\sim}{x}^T \cdot \underset{\sim}{L} = \frac{1}{\sqrt{2T}}$

Consider vectors $\underset{\sim}{r}$ such that $\underset{\sim}{r} - \underset{\sim}{x}_p \perp$ to $\underset{\sim}{L}$

$(\underset{\sim}{r} - \underset{\sim}{x}_p)^T \cdot \underset{\sim}{L} = 0 \qquad \underset{\sim}{r}^T \cdot \underset{\sim}{L} = \underset{\sim}{x}_p^T \cdot \underset{\sim}{L} = \frac{1}{\sqrt{2T}} = \text{const.}$

These vectors $\underset{\sim}{r}$ define an invariant plane; $\underset{\sim}{x}_p$ is the point where the ellipsoid is tangent to this plane.

$\underset{\sim}{r}^T \cdot \underset{\sim}{L} = \frac{1}{\sqrt{2T}}$ depends only on T, $\underset{\sim}{L}$ which are conserved.

Poinsot: See Fig 51.1

The motion of the rigid body is equivalent to the ellipsoid rolling on the invariant plane. As it rolls, $\underset{\sim}{x}_p$ moves on the plane, and on the surface of the ellipsoid

$\Rightarrow$ the direction of $\underset{\sim}{\omega}$ "precesses" around $\underset{\sim}{L}$.

For a non degenerate ellipse, $\underset{\sim}{\omega}$ will <u>not</u> lie on a circular cone, rather it will "nutate", moving closer or farther from $\underset{\sim}{L}$.

Frequency of nutation = twice frequency of precession.

The motion of the rotator can now be visualized. The center of ellipsoid is fixed and the ellipsoid rolls on a fixed plane. There three parts to the general motion: (1) rotation about the instantaneous axis, (2) precession of the instantaneous axis about the direction of $\underset{\sim}{L}$ (3) nutation

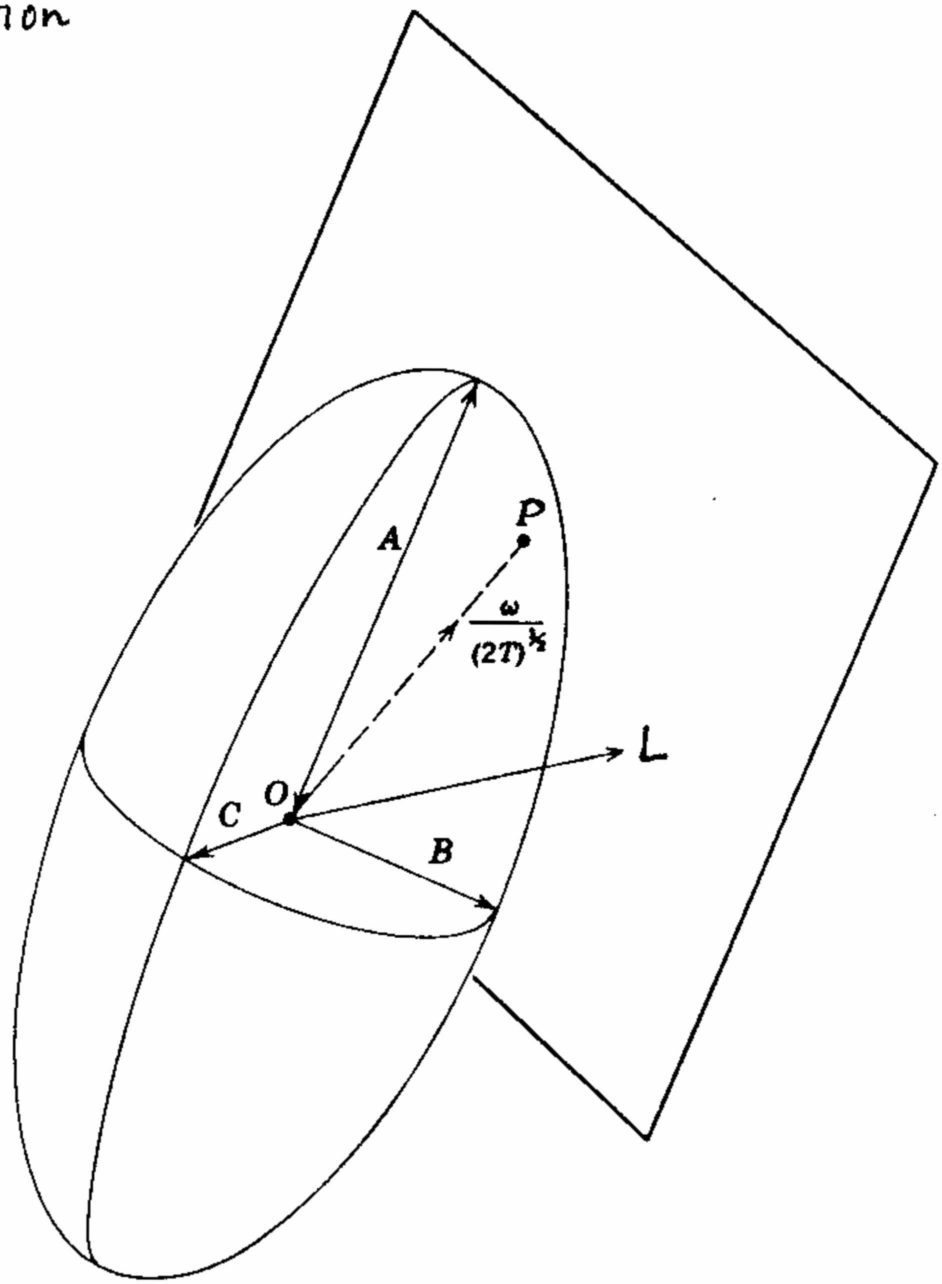


Fig. 51-1. Poinsot's representation of the motion of a free rotator. The tangent plane is at right angles to the angular momentum and the ellipsoid is in contact with it instantaneously at point P.