Brief summary of KAM theorem

Paraphrasing: For sufficiently small perturbation, almost all tori are preserved

"sufficiently small"
and "almost all" are delicate things to estimate

However tori obeying $\vec{\omega}\cdot\vec{k}=0$ (the rational tori) are destroyed. The provide the seeds of chaotic behavior
(Interesting number theory related to how closely we can approximate an irrational be a rational number eg. $\pi = 3.141592656$

$\frac{22}{7} = 3.1429\ldots$ $\frac{333}{106} = 3.14151$ $\frac{355}{113} = 3.1415929$

known to Tsu-Chung Chi in 5th China

KAM showed that preserved tori satisfy the irrationality condition

$$\left| \frac{\omega_1}{\omega_2} - \frac{r}{s} \right| > \frac{K(\epsilon)}{s^{2.5}} \quad \text{for all } r, s$$

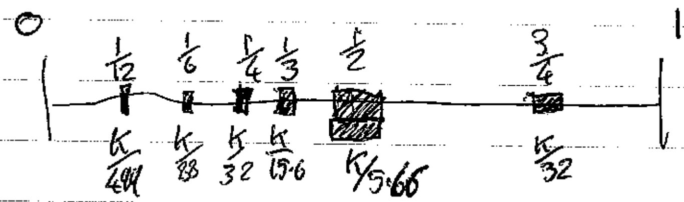
Not much is known about $K(\epsilon)$ other than $K(\epsilon) \to 0$ as $\epsilon \to 0$

The destroyed tori are the complimentary set satisfying

$$\left|\frac{\omega_1}{\omega_2} - \frac{r}{s}\right| < \frac{K(\varepsilon)}{s^{2.5}}$$

This is more restrictive than commensurability $n_1\omega_1 + n_2\omega_2 = 0$, but still allows a finite measure of preserved Tori

Consider a ~~finite~~ unit interval and delete zones of width $K/s^{2.5}$



Deleted ~~zones~~ ~~of width~~ $K/s^{2.5}$ at rational points in the unit interval

Total deleted length : # of irrationals of that denominator on unit line.

e.g. $s=3$ $\frac{1}{3}, \frac{2}{3}, \frac{3}{3}$

$s=4$ $\frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{4}{4}$

$s=5$ $\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{5}{5}$

$$\sum_{s=1}^{\infty} \frac{K}{s^{2.5}}\, s = K \sum_{s=1}^{\infty} \frac{1}{s^{1.5}} \simeq K$$

(crude over-estimate)

this $\rightarrow 0$ as $\varepsilon \rightarrow 0$

In fact any width $\frac{K}{s^{u}}$ with $u > 2$ would ensure a finite measure of preserved tori.