Integrable Hamiltonians

Saw previously that separable systems can be integrated to give n constants of the motion $\alpha_1, \ldots \alpha_n$, but systems are not generally separable.
The key ~~is~~ to integrating a Hamiltonian system is to find n independent "integrals of motion" (constants).
If these can be expressed as n constant conjugate momenta then we can trivially integrate Hamilton's equations.

Furthermore, if we can find n independent "librational" paths C_k as set of action variables can be defined explicitly.
In fact, this does <u>not</u> require separability of the form $S = \sum_{k=1} S_k(q_k, \alpha_1 \ldots \alpha_n)$.

What we need are n independent functions F_i $(i = 1, \ldots n)$ of the variables $q_1 \ldots q_n, p_1 \ldots p_n$ which are constants of the motion

$$F_i(\vec{p}(t), \vec{q}(t)) = f_i$$

i.e. same number of symmetries as there are degrees of freedom. Then system is <u>integrable</u>. (otherwise non-integrable)

Recall Poisson bracket:

$$\frac{dF_i}{dt} = [F_i, H] + \underbrace{\frac{\partial F_i}{\partial t}}_{=0}$$

confine discussion to autonomous systems

$\Rightarrow$ n Poisson brackets $[F_i, H] = 0$

Can ~~also~~ take H as one of the F_i
(invariance of H corresponds to symmetry wrt time translation.

In fact we also require

$$[F_i, F_j] = 0 \qquad i,j = 1, \dots n.$$

the functions F_i are "in involution"

Each F_i = constant confines a trajectory to a $(2n-1)$-dimensional subspace of the full $2n$-dim phase space.

$\Rightarrow$ n conditions $\Rightarrow$ n-dim intersection (manifold) M of these subspaces

The coords and momenta at $t=0$ determine the n values of the F_i and hence a particular M on which trajectories are confined

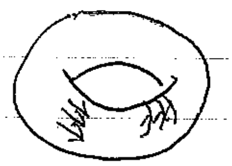
Normal to manifold is given by $2n$-component gradient vector $(\vec{\nabla}_q F_i, \vec{\nabla}_p F_i)$

The Poisson bracket condition is a scalar product expressing orthogonality between the gradient vector and members of a set of n vector fields

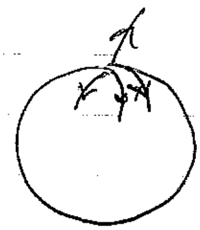
$$\vec{V}_j \doteq (\nabla_p F_j, -\nabla_q F_j) \qquad j = 1, \dots n$$

$\vec{V}_j$ at a particular point are all tangent to surface M which contains that pt.

n vector fields which are linearly independent on an n-dim surface (define a coord grid without a singularity)
$\Rightarrow$ by Poincaré - Hopf thm (Hairy ball thm) M has topology of an n-torus



smoothly combed vector fields on a 2-torus



singular pt on a 2-sphere

Amazing! tori in phase space provide means of defining action variable in an invariant (representation independent) way.

n-torus is naturally periodic : n independent 2π periodicities i.e. n topologically independent closed paths C_k. None can be shrunk to zero or continuously deformed to another.

$$\Rightarrow \qquad I_k = \frac{1}{2\pi} \oint_{C_k} \sum_{m=1}^{n} p_m \, dq_m$$

$$\theta_k = \frac{\partial}{\partial I_k} S(q_1 \dots q_n, I_1 \dots I_n)$$

If system is integrable the transformation to action angle variables is global ! whole phase space is filled with tori

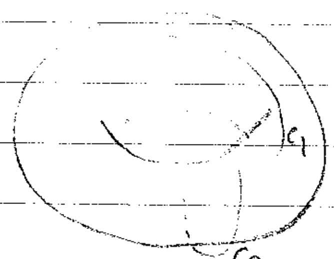
Particular set of initial conditions : $(q_1(0) \dots q_n(0), p_1(0) \dots p_n(0))$ fixes $f_i = F_i(\vec{p}(0), \vec{q}(0))$. Set F_i determines on which torus the trajectory lies (i.e. actual values of I_i, $i = 1, \dots n$). Value of θ_i at given time determines position on torus.

Degrees of freedom n	1	2	3	4
Dim of phase space	2	4	6	8
Dim of energy shell	1	3	5	7
Dim of Tori	1	2	3	4

N.B. 1. When $n=1$, the energy shell and torus are same thing
$\Rightarrow$ system is ergodic i.e. trajectory uniformly explores the energy shell, so that a time average is the same as a phase average

$$\lim_{T\to\infty}\int_{-T}^{T}\underbrace{f(p(t),q(t))}_{\text{quantity we wish to average}}dt = \iint dp\,dq\ \ f(p,q)\ \delta(E-H(p,q))$$

2.



Examples of integrable systems in 2D

(1) 2D harmonic oscillator $\qquad H=\frac{1}{2}\left(p_1^2+p_2^2+\omega_1^2q_1^2+\omega_2^2q_2^2\right)$

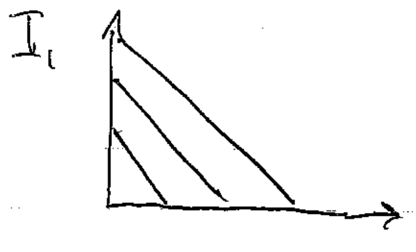
two integrals of motion (energies associated with each mode)

$$F_1=E_1=\tfrac{1}{2}\left(p_1^2+\omega_1^2q_1^2\right),\quad F_2=E_2=\tfrac{1}{2}\left(p_2^2+\omega_2^2q_2^2\right)$$

This system has a separable HJ equation $\Rightarrow$ independent librations in the (p_1,q_1) and (p_2,q_2) planes.

Actions : $I_1 = \frac{1}{2\pi}\oint_{C_1} p_1(q_1, E_1, E_2)\, dq_1$ $\qquad I_2 = \frac{1}{2\pi}\oint_{C_2} p_2(q_2, E_1, E_2)\, dq_2$

$$H(I_1, I_2) = I_1\omega_1 + I_2\omega_2$$



Contours of const. $E = H(I_1, I_2)$ in the "action plane". Each pt on a contour corresponds to a particular torus in phase space

(2). Free particle in 2D box $\quad H = \frac{1}{2}(p_x^2 + p_y^2) \qquad 0 < x < a,\ 0 < y < b$

Moduli of momenta are constants of motion $\quad$ ($p_x \to -p_x$ on collision with a wall)

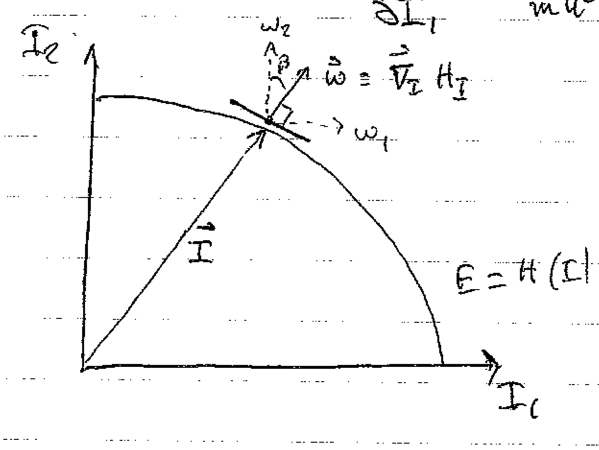
$$I_1 = \frac{1}{2\pi}\int_{C_x} p_x\, dx = \frac{a}{\pi}|p_x|$$

$$I_2 = \frac{1}{2\pi}\int_{C_y} p_y\, dy = \frac{b}{\pi}|p_y|$$

and $\quad H(I_1, I_2) = \frac{\pi^2}{2m}\left(\frac{I_1^2}{a^2} + \frac{I_2^2}{b^2}\right)$

N.B. this is a non-linear system since frequencies are action-dependent.

$$\omega_1 = \frac{\partial H}{\partial I_1} = \frac{\pi^2}{ma^2} I_1 \qquad \omega_2 = \frac{\partial H}{\partial I_2} = \frac{\pi^2}{mb^2} I_2$$



Vector $\vec{I}$ identifies a particular torus on the energy surface.
Normal to surface $\vec{\omega}$ determines the frequency vector, Frequencies vary from torus to torus.

$$\vec{\nabla}_I = (\partial I_1, \partial I_2)$$

3 Motion on plane under central force

$$H = \frac{p_r^2}{2m} + \frac{p_\phi^2}{2mr^2} + V(r) \qquad (r, \phi) \text{ plane polar coords}$$

p_ϕ = ang. mom = conserved

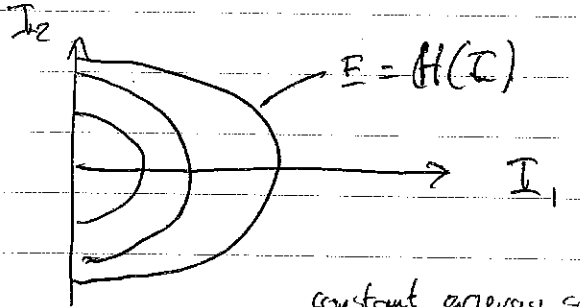
$$F_1 = p_\phi \quad , \quad F_2 = H(p_r, p_\phi, r) = E$$

$$I_1 = \frac{1}{2\pi}\oint p_\phi \, d\phi = p_\phi$$

$$I_2 = \frac{1}{2\pi}\oint p_r \, dr = \frac{1}{\pi}\int_{r_1}^{r_2} \sqrt{2m\left(E - \frac{I_1^2}{2mr^2} - V(r)\right)} \, dr$$

Precise form depends on $V(r)$

r_1 & r_2 are the librational turning pts.



constant energy surfaces

Motion on Tori

~~Motion~~ Tori are periodic $\Rightarrow$ can express a given dynamical quantity $f(\vec{p}, \vec{q})$ as a multiple Fourier series in the angle variables $\theta_1, \theta_2 \ldots \theta_n$

the k's are integers

e.g. $$q_i(t) = \sum_{k_1=-\infty}^{+\infty} \cdots \sum_{k_n=-\infty}^{+\infty} a^{(i)}_{k_1 k_2 \ldots k_n} e^{i(k_1\theta_1 + k_2\theta_2 + \ldots k_n\theta_n)}$$

$$= \sum_{k_1} \cdots \sum_{k_n} a^{(i)}_{k_1, \ldots k_n} e^{i(k_1\omega_1 + \ldots k_n\omega_n)t + i(k_1\delta_1 + \cdots k_n\delta_n)}$$

Fourier coefficients :

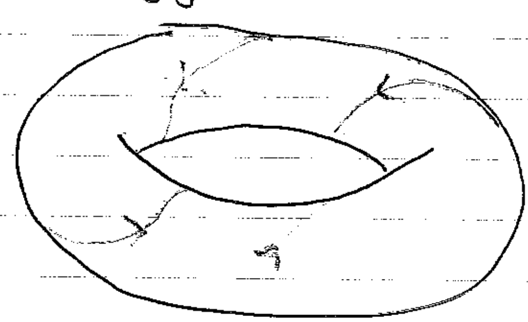
$$a^{(i)}_{k_1,\ldots k_n}$$

$$a^{(i)}_{\vec{k}}(\vec{I}) = \int_0^{2\pi} d\theta_1 \ldots \int_0^{2\pi} d\theta_n \; q_i(\vec{I},\vec{\theta})\; e^{-i(k_1\theta_1 + \cdots k_n\theta_n)}$$

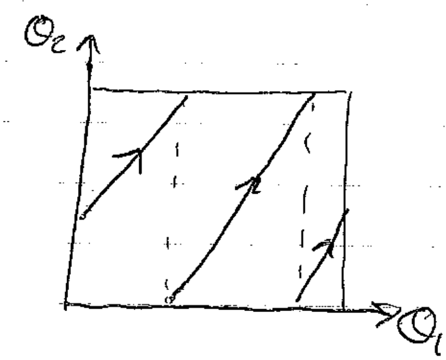
If the frequencies ω_i are rationally related

e.g. for $n=2$ $\qquad \dfrac{\omega_1}{\omega_2} = \dfrac{n}{m}$

the orbit will close after m cycles of θ_1 and n of θ_2
Terminology : "closed orbits".



$\Longrightarrow$ topologically equivalent to square with edges identified



n-dimensional condition : $\sum_{i=1}^{n} k_i \omega_i = 0$

(e.g. for $n=2$ we might have $k_1 = n$, $k_2 = m$
$n\,\omega_1 \mp m\,\omega_2 = 0 \Rightarrow \dfrac{\omega_1}{\omega_2} = \dfrac{m}{n}$)

If the frequencies are not rationally related the motion on a given torus will never exactly repeat itself $\Rightarrow$ terminology "quasi-periodic orbits". In this case a single orbit will cover the torus uniformly. Terminology : "flow is ergodic on the torus"
[proved by Nicholas Oresme ca (1325-1382) and more rigourously by Jacobi in 1835]

Rational or irrational, the motion is still integrable.

A completely integrable system is called <u>non-degenerate</u> if

$$\det\left|\frac{\partial \omega_i(I)}{\partial I_j}\right| = \det\left|\frac{\partial^2 H(I)}{\partial I_i \partial I_j}\right| \neq 0$$

This ensures frequencies vary from torus to torus
ie. system is nonlinear (counter example: SHO)
Thus on a given energy shell some tori will be covered ~~with~~ by closed orbits and others by quasi-periodic orbits.

There are infinitely many rational numbers but they form a set of measure zero in the space of real numbers ie they are infinitely outnumbered by the irrationals.

<u>Important questions</u>

1) What happens if system does not possess n integrals of motion?

2) Given a Hamiltonian, how do we know if the system is integrable?

<u>Elaboration</u>

~~Answers~~ 1). If $H = \underbrace{H_0}_{\text{integrable}} + \underbrace{\varepsilon H_1}_{\text{small perturbation}}$ are the tori preserved (albeit distorted)?

- big question of 19th and 20th mechanics. Answered in 1960s by the Kolmogorov, ~~Moser~~ Arnold & Moser (KAM) theory

2). Can try and find integrals but generally an impossible task
$\Rightarrow$ still a topic of current research.