Ignorable coordinates and constants of motion

$$L(q,\dot q,t) = T(q,\dot q,t) - U(q,t)$$

Conservative forces + holonomic constraints

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_j}\right) = \frac{\partial L}{\partial q_j}$$

$$\frac{\partial L}{\partial \dot q_j} = p_j$$ canonical momentum or conjugate momentum or generalized momentum

If q_r does not occur in L, then $\frac{\partial L}{\partial \dot q_j} = \text{const}$

e.g. particle moving ch a plane under a central conservative potential $\Rightarrow$ ~~~~ L does not depend on ϕ

$\Rightarrow$ Conservative of angular momentum

$$L = \tfrac{1}{2} m(\dot r^2 + r^2\dot\phi^2) - U(r)$$

$$p_r = \frac{\partial L}{\partial \dot r} = m\dot r \qquad \frac{\partial L}{\partial r} = m r \dot\phi^2 - \frac{\partial U}{\partial r}$$

$$p_\phi = \frac{\partial L}{\partial \dot\phi} = m r^2 \dot\phi = \text{const.} = J$$

Whenever a system exhibits a continuous symmetry there will be a corresponding conserved quantity (Emmy Noether 1918)

Lagrangian is symmetric under:	Conserved quantity
translations in space	linear momentum
rotations	angular momentum
time	energy

Hamiltonian as a Legendre transform of L

Lagrangian mechanics $L(q, \dot{q}, t)$

Qu: Can we have (q, p, t)?

Ans: Yes, $H(q, p, t)$.

Lagrange's equs: $\dot{p}_j = \frac{\partial L}{\partial q_j}$

where $p_j = \frac{\partial L}{\partial \dot{q}_j}$ generalized momentum

2n variables $\{q_1, \dots q_n; \dot{q}_1, \dots \dot{q}_n\}$

However can solve $p_j = \frac{\partial L}{\partial \dot{q}_j}$ for $\dot{q}$ in terms of q and p.

gives $\dot{q}_j = \dot{q}_j(q_1, \dots q_n; p_1 \dots p_n)$ 2n variables

e.g. particle in a plane in polar coords:

$$p_r = m\dot{r} \qquad p_\theta = mr^2\dot{\theta}$$

$$\Rightarrow \dot{r} = \frac{p_r}{m}, \quad \dot{\theta} = \frac{p_\theta}{mr^2}$$

Coords: $(r, \theta; p_r, p_\theta)$

Define

$$H(q,p) = \sum_j p_j \dot{q}_j(q,p) - L(q, \dot{q}(q,p))$$

"Consider derivative wrt p":

$$\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha + \underbrace{\sum_j p_j \frac{\partial \dot{q}_j}{\partial p_\alpha} - \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial p_\alpha}}_{=0\ !}$$

$$\Rightarrow \quad \boxed{\frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha}$$

"Now consider derivative wrt q"

$$\frac{\partial H}{\partial q_\alpha} = -\frac{\partial L}{\partial q_\alpha} + \underbrace{\sum_j p_j \frac{\partial \dot{q}_j}{\partial q_\alpha} - \sum_j \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial q_\alpha}}_{=0\ !}$$

but, $\frac{\partial L}{\partial q_\alpha} = \dot{p}_\alpha$ by Lagrange eqn

$$\Rightarrow \quad \boxed{\frac{\partial H}{\partial q_\alpha} = -\dot{p}_\alpha}$$

Hamilton's eqns.

Lagrange : n 2nd-order differential eqns

Hamilton : $2n$ 1st-order " "

Can prove : 1) If $L(q, \dot{q})$ has no explicit t dependence $\left(\frac{\partial L}{\partial t} = 0\right)$ neither does H.

2) When $\frac{dH}{dt} = 0$, H = total energy

(see p 280 in Kibble & Berkshire, 5th Edition)

"Demonstration" In a time independent system

$$T = \sum_j c_j \dot{q}_j^{\,2}$$ homogeneous quadratic function of $\dot{q}$ (see p 233 K & B 5th Ed)

then $$\sum_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} \dot{q}_\alpha = 2T$$

but $$p_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha}$$ (conservative force)

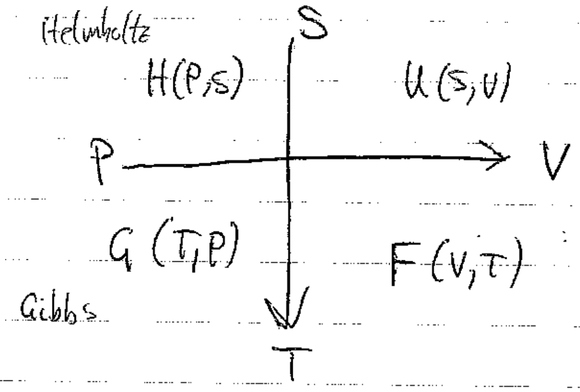
then $$H = \sum_j p_j \dot{q}_j - L = \sum_j \frac{\partial T}{\partial \dot{q}_j} \dot{q}_j - (T - V)$$

$$= 2T - (T - V) = T + V$$

total energy of system.

Legendre or contact transform

Four thermodynamic potentials



$$dU = TdS - PdV \qquad T = \left(\frac{\partial U}{\partial S}\right)_V$$

$$F \equiv -\underset{\text{new}}{T}\underset{\substack{\text{old} \\ \text{variable}}}{S} + U(V,S)$$

$$dF = -TdS - SdT + (TdS - PdV)$$

$$= -SdT - PdV$$

so $F = F(V,T)$

i.e. $$F(V,T) = U(V,S) - S\underbrace{\left(\frac{\partial U}{\partial S}\right)_V}_{=T}$$

$$-H(q,p) = L(q,\dot q) - \dot q\left(\frac{\partial L}{\partial \dot q}\right)_q$$

choice of sign is a convention.