Hamilton – Jacobi eqn

Find canonical transformation to give constant conjugate momenta.

$$\text{Choose} \quad F_2(\underbrace{q_1 \ldots q_n}_{\text{old coord}}, \underbrace{P_1 \ldots P_n}_{\text{new momenta}}) \equiv S(q_1 \ldots q_n ; \underbrace{\alpha_1 \ldots \alpha_n}_{\substack{\text{convention} \\ \text{- new momenta}}})$$

$$p_i = \frac{\partial}{\partial q_i} S(q_1 \ldots q_n, \alpha_1 \ldots \alpha_n)$$

$$\beta_i = \frac{\partial}{\partial \alpha_i} S(q_1 \ldots q_n, \alpha_1 \ldots \alpha_n)$$

new coords conjugate to α_i

Also
$$\underset{\uparrow \text{old}}{H}\left(q_1, \ldots q_n, \frac{\partial S}{\partial q_1}, \ldots \frac{\partial S}{\partial q_n}\right) = \underset{\uparrow \text{new}}{H'}(\alpha_1, \ldots \alpha_n)$$

RHS is a constant quantity.

This is a 1st order partial differential eqn for S in n independent variables $(q_i, i = 1, \ldots n)$ ~~t~~ $\Rightarrow$ time-independent Hamilton-Jacobi equation.

N.B. time-dependent version

$$\frac{\partial S}{\partial t} + H\left(q_1, \ldots q_n, \frac{\partial S}{\partial q_1} \ldots \frac{\partial S}{\partial q_n}, t\right) = 0$$

For conservative systems time dependence separates

$$S = S(q_1 \ldots q_n, \alpha_1 \ldots \alpha_n) - Et$$

and we return to $\quad E = H'(\alpha_1 \ldots \alpha_n)$

Complete solution : n independent constants of integrations i.e. the $\alpha_1, \alpha_2 \ldots \alpha_n$

solving HJ eqns $\Longleftrightarrow$ solving canonical eqns of motion

<u>Form of solution</u> (for a fixed set of α_i, i.e. $d\alpha_i = 0$)

$$dS = \sum_{i=1}^{n} \frac{\partial S}{\partial q_i} dq_i = \sum_{i=1}^{n} p_i \, dq_i$$

$$\Rightarrow \quad S = \int_{q_0}^{q} \sum_{i=1}^{n} p_i \, dq_i \qquad \text{line integral}$$

$\vec{q}_0 = q_1(0) \ldots q_n(0)$ initial pt on classical trajectory with a given set of α_i

$\vec{q} = q_1(t) \ldots q_n(t)$ is a moving pt on the path

$\Rightarrow$ Need to know path (solution of problem)

<u>1 degree of freedom</u> Trick : set $\alpha = H'$ (can choose it as we wish)

$$H\left(q, \frac{\partial S(q,\alpha)}{\partial q}\right) = \alpha$$

then α = energy of system

$$p = \frac{\partial}{\partial q} S(q,\alpha)$$

$$\beta = \frac{\partial}{\partial \alpha} S(q,\alpha)$$

transformation is canonical, so
$$\dot{\alpha} = -\frac{\partial H'}{\partial \beta} = 0$$

$$\dot{\beta} = \frac{\partial H'}{\partial \alpha} = 1$$

Integrate : $\alpha = \text{constant} = H'$

$$\beta = t - t_0$$

Now ~~that~~ $\beta = \frac{\partial}{\partial \alpha} S$, $S = \int_{q_0}^{q} p \, dq$

so $$t - t_0 = \frac{\partial}{\partial \alpha} \int_{q_0}^{q} p(q, \alpha) \, dq = \int_{q_0}^{q} \frac{\partial}{\partial \alpha} p(q, \alpha) \, dq$$

For a "standard" Hamiltonian

$$H(p, q) = \frac{p^2}{2m} + V(q).$$

then $$H(p, q) = H'(\alpha) = \alpha$$

So $$p(q, \alpha) = \pm \sqrt{2m(\alpha - V(q))}$$

so $$t - t_0 = \sqrt{\frac{m}{2}} \int_{q_0}^{q} \frac{dq}{\sqrt{\alpha - V(q)}}$$

Reduced problem to "quadrature"

Can solve to give $q(t)$.

Action-Angle variables

non-chaotic

[illegible] for [illegible] systems [illegible]

For bounded systems ~~If energy is conserved~~, phase space trajectories are closed curves $\Rightarrow$ motion is periodic, returning to same (p, q) point after period $\frac{2\pi}{\omega}$

Idea of Action-Angle variables : find pair of canonically conjugate variables (I, θ) so that coordinate θ increases by 2π after each complete period of motion. $I = \text{constant}$

$$p = \frac{\partial}{\partial q} S(q, I)$$

$$\Theta = \frac{\partial}{\partial I} S(q, I).$$

HJ eqn $\quad H\left(q, \frac{\partial S}{\partial q}\right) = \alpha = H'(I)$

For a path with a fixed value of α (and hence I)

$$\frac{\partial \Theta}{\partial q} = \frac{\partial}{\partial I} \frac{\partial S}{\partial q}$$

Now we require that over one cycle around invariant curve C, (i.e. fixed value of α), change in Θ is 2π

$$2\pi = \oint_C d\Theta = \frac{\partial}{\partial I} \oint_C \frac{\partial S}{\partial q} dq = \frac{\partial}{\partial I} \oint_C p\, dq$$

This condition can be satisfied if

$$\boxed{I = \frac{1}{2\pi} \oint_C p\, dq} \qquad \text{Action variable}$$

~~how do we do this integral?~~

~~The HJ eqn is still solved for S as a fn of q and α~~

$$\cancel{S = \int_{q_0}^{q} p(q, \alpha)\, dq}$$

~~then~~ Canonical eqns of motion: $\dot{I} = -\frac{\partial}{\partial \Theta} H'(\Theta) = 0$

$$\dot{\Theta} = \frac{\partial}{\partial I} H'(I) = \omega(I)$$

<u>Integrate</u> $\quad I = \text{const}$

$$\Theta = \omega(I)t + \delta$$

To obtain $p = p(I, \theta)$ $\quad q = q(I, \theta)$

(1) $I = \frac{1}{2\pi}\int_c p(q,\alpha)\,dq$ gives connection between I and α
i.e. a given α gives an I.

(2) In $S = \int_{q_0}^{q} p(q,\alpha)\,dq$ replace α by I
$\Rightarrow S = \int_{q_0}^{q} p(q,I)\,dq$

(3) Use $\left\{ \begin{array}{l} p = \frac{\partial}{\partial q} S(q,I) \\ \theta = \frac{\partial}{\partial I} S(q,I) \end{array} \right\}$ to obtain explicit dependence of p and q on I and θ

<u>Example</u> SHO

$$H = \tfrac{1}{2}(p^2 + \omega^2 q^2) \qquad \text{i.e. } m=1$$

HJ eqn

$$\tfrac{1}{2}\left(\frac{\partial S}{\partial q}\right)^2 + \tfrac{1}{2}\omega^2 q^2 = \underset{=E}{\alpha}$$

$$I = \frac{1}{2\pi}\oint_c \sqrt{2(E - \tfrac{1}{2}\omega^2 q^2)}\,dq.$$

round trip between turning pts at $q = \pm\frac{\sqrt{2E}}{\omega}$

Evaluate integral: $I = \frac{E}{\omega}$

ellipse $1 = \frac{p^2}{2E} + \frac{\omega^2 q^2}{2E}$

$a = \frac{\sqrt{2E}}{\omega}$ $\quad b = \sqrt{2E}$

Area $= \pi ab = \pi \frac{2E}{\omega}$

but $I = \frac{1}{2\pi}$ Area

$\Rightarrow I = \frac{E}{\omega}$

This gives relation between $\alpha = E$ and new momentum I

$$\alpha = E = H'(I) = I\omega$$

Generating function: $S(q,I) = \int_{q_0}^{q} \sqrt{2(I\omega - \tfrac{1}{2}\omega^2 q^2)}\,dq$

from $\Theta = \frac{\partial}{\partial I} S(q, I)$ one obtains $q = q(I, \Theta)$

we get $q = \sqrt{\frac{2I}{\omega}} \sin(\Theta + \delta)$

δ = some phase = $\sin^{-1}\left(\frac{q_0 \omega}{\sqrt{2E}}\right)$

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i.e. $\theta = \frac{\partial S}{\partial I} = \int_{q_0}^{q} \frac{\frac{1}{2} 2\omega}{\sqrt{2(I\omega - \cancel{\frac{1}{2}}\omega^2 q^2)}} dq = \frac{\omega}{\sqrt{2I\omega}} \int_{q_0}^{q} \frac{1}{\sqrt{1 - \frac{\omega^2}{I\omega}\frac{q^2}{2}}} dq$

Now $\arcsin x = \int_0^x \frac{1}{\sqrt{1-z^2}} dz \qquad |x| \le 1$

put $\frac{\omega}{2I} q^2 = z^2 \qquad \Rightarrow q = \sqrt{2}\sqrt{\frac{I}{\omega}} z$

$\Theta = \frac{\omega}{\sqrt{2I\omega}} \int_{q_0}^{q \to \sqrt{\frac{2I}{\omega}} z} \frac{1}{\sqrt{1-z^2}} dz \left(\sqrt{2}\sqrt{\frac{I}{\omega}}\right)$

i.e. $\int_{q_0}^{q} = \int_0^{q} - \int_0^{q_0}$

$= \operatorname{Arcsin} \sqrt{\frac{2\omega}{I}}\, q - \underbrace{\operatorname{Arcsin} \sqrt{\frac{2\omega}{I}}\, q_0}_{\equiv \delta}$

Invert: $\sin[\Theta + \delta] = \sqrt{\frac{2\omega}{I}}\, q$, or $q = \sqrt{\frac{I}{2\omega}} \sin(\Theta + \delta)$.

Hamilton's equations! $\dot{\Theta} = \frac{\partial H}{\partial I}$

but $E = I\omega \quad \Rightarrow \quad \dot{\Theta} = \omega$

$\Rightarrow \quad \Theta = \omega t + \delta$