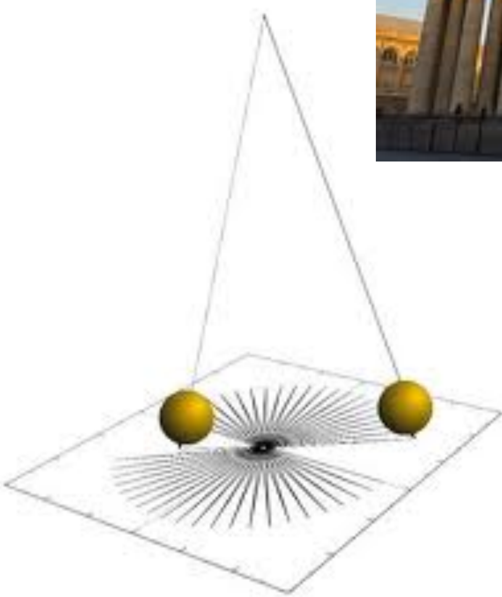
Foucault’s Pendulum (1851)

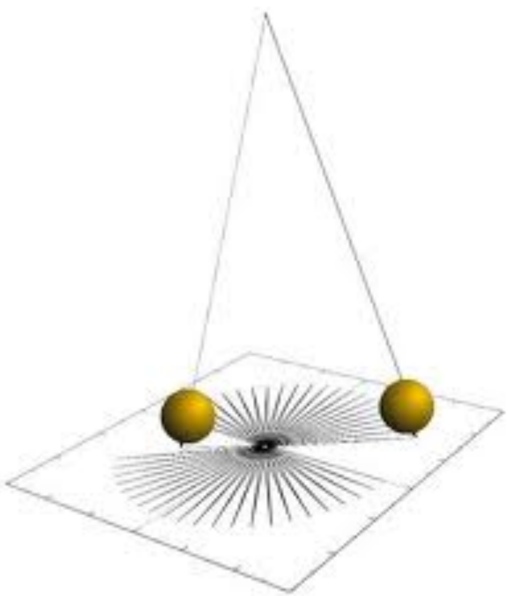
Pantheon, Paris

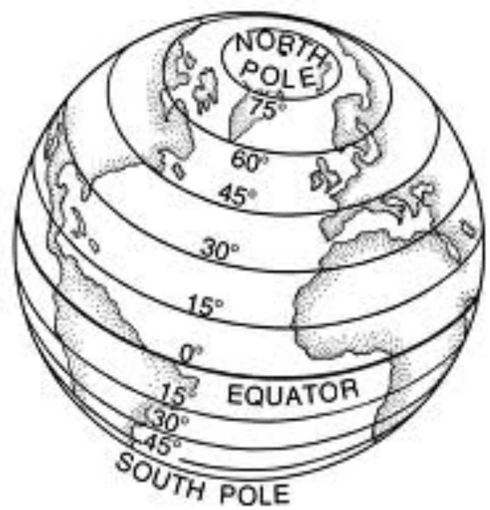


- during one rotation of the Earth (24 hours) the plane of oscillation rotates by angle:

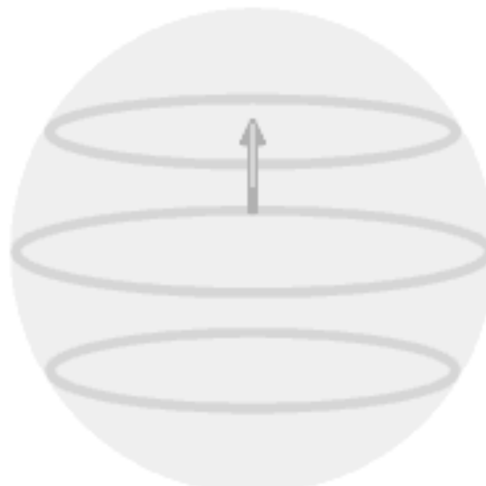
$$\alpha = -2\pi \sin\theta$$

$\theta = \text{latitude (measured from equator)}$

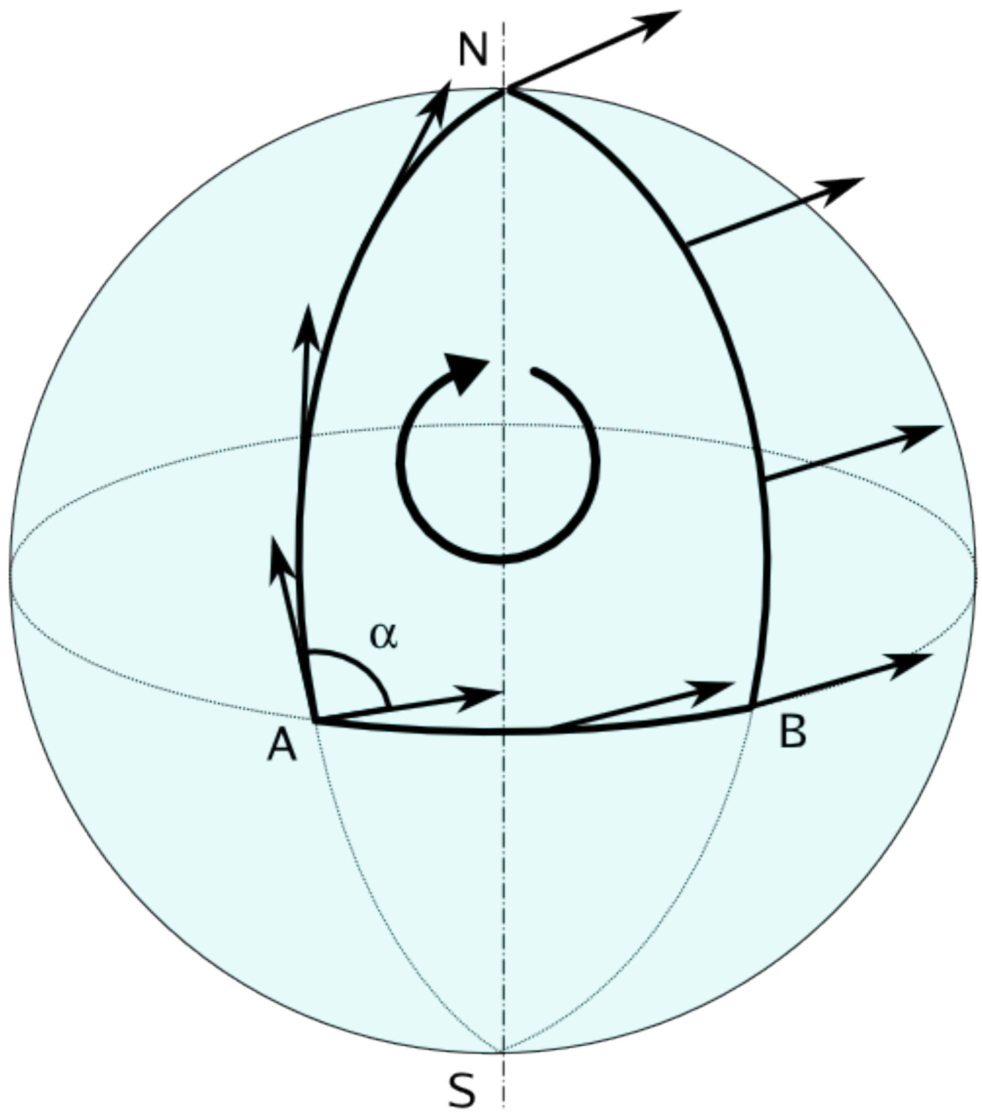




Rotation of plane of pendulum swing



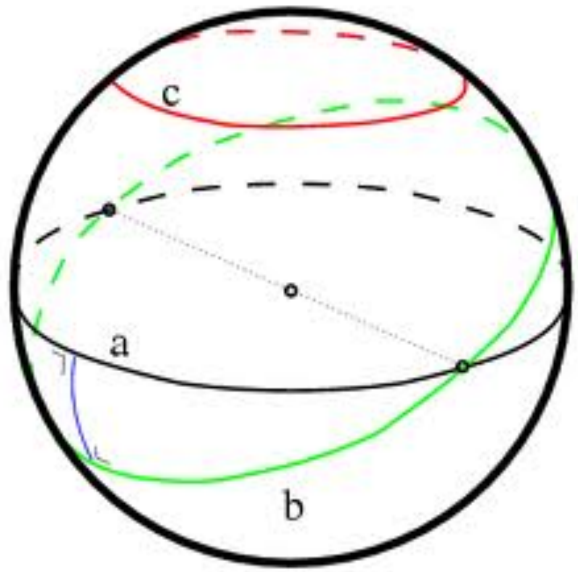
Parallel transport of a vector around a circuit on a curved surface



Parallel transport: vector is not allowed to rotate about local normal to surface.

The change in orientation of the vector after completing the circuit is an example of **anholonomy** (sometimes called holonomy!)

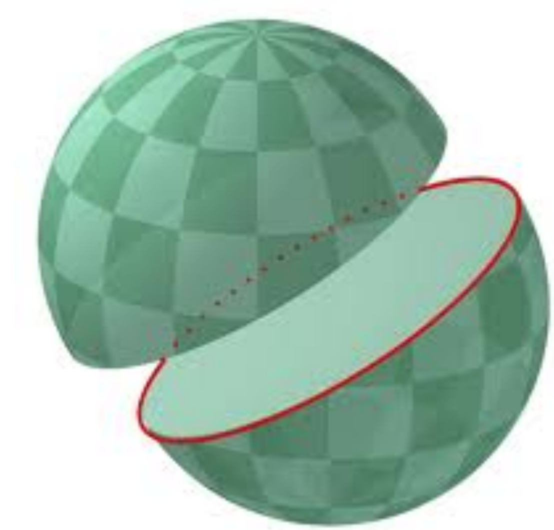
Great circles



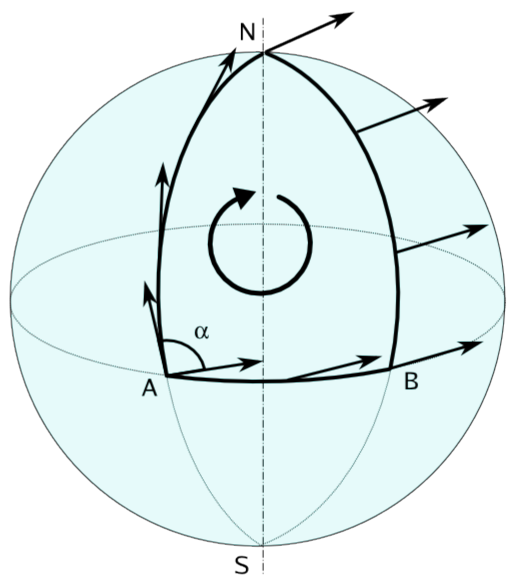
Great circles give “straight lines” on a sphere, i.e. shortest distance between two points. They are therefore geodesics (c.f. Newton’s 1st law)

Great circles

- They cut sphere exactly in half
- Have same radius as sphere
- Pass through centre of sphere



Parallel transport along great circles

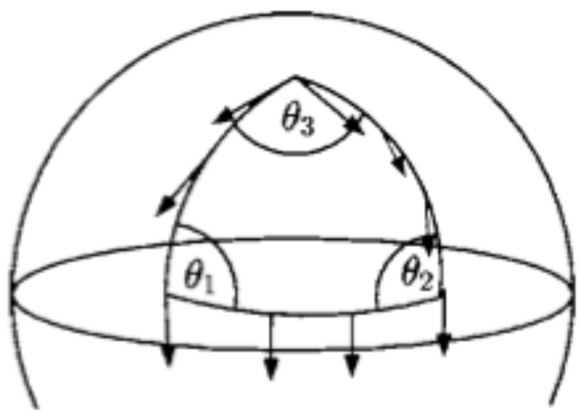


- Moving along a great circle on a sphere is like moving along a straight line in flat space: the plane of oscillation of pendulum does not rotate.
- Why? Because the two halves of the sphere have equal mass and so there is nothing to break the symmetry.
- Any path can be broken up into sum of many infinitesimal great circles.

Gauss-Bonnet theorem

$$\alpha = \frac{S}{r^2}$$

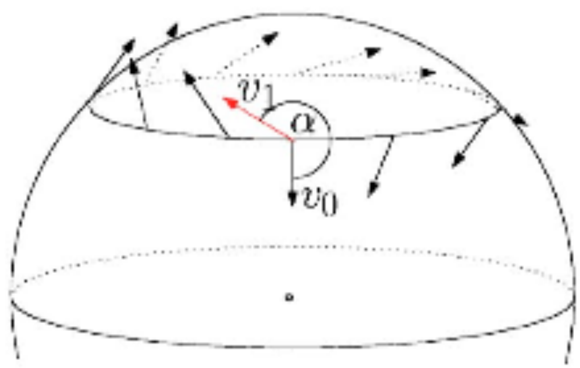
$$S = \text{area on surface of sphere enclosed by path}$$

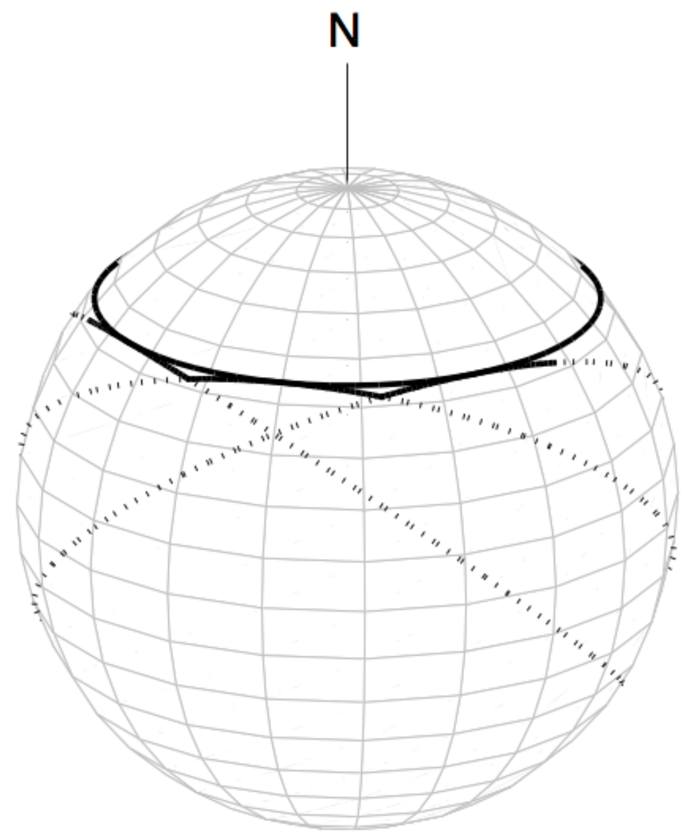


$$\alpha = (\theta_1 - \pi) + (\theta_2 - \pi) + (\theta_3 - \pi) = \theta_1 + \theta_2 + \theta_3 - \pi$$

Gauss proved that: $S = (\theta_1 + \theta_2 + \theta_3 - \pi) r^2$ this would be zero on a flat surface!

A path following a line of latitude is not a great circle, but can broken up into short pieces, each one of which is part of a great circle:

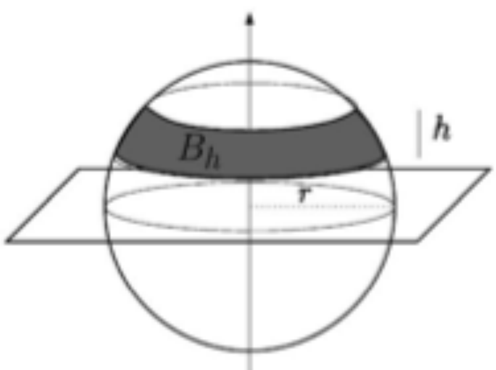




Area enclosed by path: $$S = 2\pi r^2(1 - \sin[\theta])$$ (see below)

Then:

$$\alpha = \frac{S}{r^2} = 2\pi(1 - \sin[\theta]) = -2\pi \sin[\theta] \quad \text{modulo} \quad 2\pi$$



Area on sphere of belt of vertical height h is the same as that of a cylinder of radius r and height h:

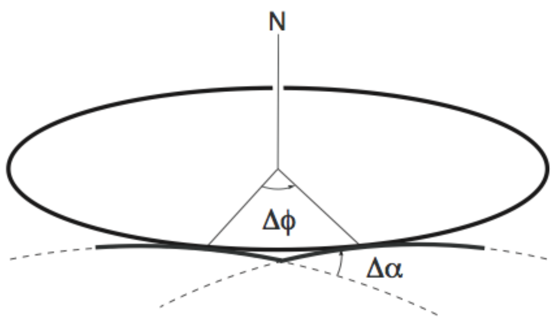
$$S_{\text{belt}} = 2\pi r h$$

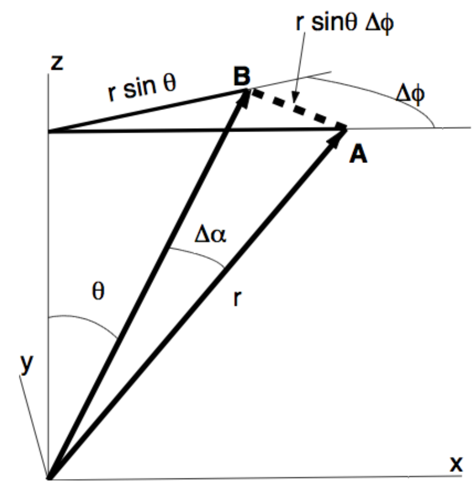
(Archimedes's grave stone!)

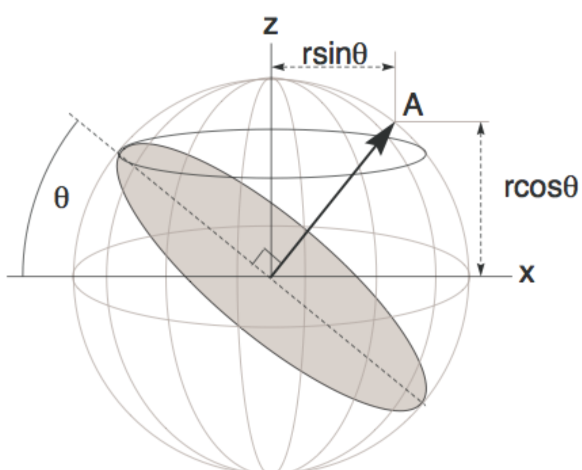
Here: $$h = r - r\sin[\theta]$$

Alternative explanation without Gauss-Bonnet theorem

Angle between two great circles that reach the same latitude, but are displaced in longitude by a small angle $\Delta\phi$ is $\Delta\alpha = \Delta\phi \sin[\theta]$







Then put

$$\Delta\phi \to 2\pi$$
$$\Delta\alpha \to \alpha$$

for a full day.