Analysis of fixed points of maps

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = T \begin{bmatrix} x_i \\ y_i \end{bmatrix}$$

Assume there is a fixed pt at $(x,y) = (0,0)$. To understand the nature of the fixed pt we linearize about it to give

$$\begin{bmatrix} \delta x_{i+1} \\ \delta y_{i+1} \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \delta x_i \\ \delta y_i \end{bmatrix}$$

i.e. $x_{i+1} = f(x_i, y_i)$ $\Rightarrow$ $\delta x_{i+1} = f(\delta x_i, \delta y_i)$

$y_{i+1} = g(x_i, y_i)$ $\quad$ $\delta y_{i+1} = f(\delta x_i, \delta y_i)$

$$f(\delta x_i, \delta y_i) = f(0,0) + \left.\frac{\partial f}{\partial x}\right|_{x=y=0} \delta x_i + \left.\frac{\partial f}{\partial y}\right|_{x=y=0} \delta y_i$$

$$g(\delta x_i, \delta y_i) = g(0,0) + \left.\frac{\partial g}{\partial x}\right|_{x=y=0} \delta x_i + \left.\frac{\partial g}{\partial y}\right|_{x=y=0} \delta y_i$$

For a fixed pt $f(0,0) = g(0,0) = 0$

$$\Rightarrow T_{11} = \left.\frac{\partial f}{\partial x}\right|_{x=y=0} \qquad T_{12} = \left.\frac{\partial f}{\partial y}\right|_{x=y=0}$$

$$T_{21} = \left.\frac{\partial g}{\partial x}\right|_{x=y=0} \qquad T_{22} = \left.\frac{\partial g}{\partial y}\right|_{x=y=0}$$

The nature of the fixed pt is determined by the eigenvalues

$$\begin{vmatrix} T_{11}-\lambda & T_{12} \\ T_{21} & T_{22}-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - \lambda(T_{11}+T_{22}) + T_{11}T_{22} - T_{12}T_{21} = 0$$

or $$\lambda^2 - \lambda \,\mathrm{trace}(T) + \det(T) = 0$$

If T is area preserving: $\det(T) = 1$

$\Rightarrow$ $$\lambda_{\pm} = \tfrac{1}{2}\,\mathrm{trace}(T) \pm \tfrac{1}{2}\sqrt{\mathrm{trace}(T)^2 - 4}$$

Note: $\lambda_+ \lambda_- = 1$

<u>3 different cases</u>

i) $|\mathrm{trace}(T)| < 2$ $\Rightarrow$ λ_+ and λ_- are complex conjugates lying on unit circle.

ie. $\lambda_+ = r e^{i\alpha} = e^{i\alpha}$
$\lambda_- = r e^{-i\alpha} = e^{-i\alpha}$.

but $r=1$ because $\lambda_+\lambda_- = 1$

2) $|\mathrm{trace}(T)| > 2$ λ_+ and λ_- are real

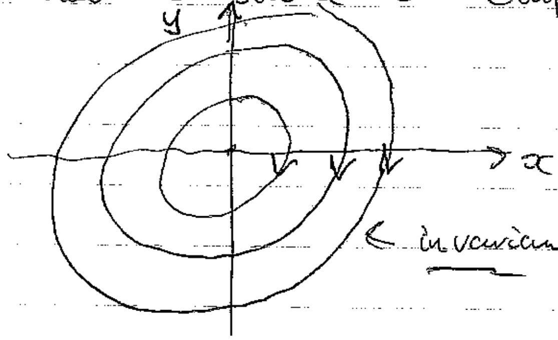
3) $|\mathrm{trace}(T)| = 2$ $\lambda_+ = \lambda_- = \pm 1$

Eigenvectors

1) $$\begin{bmatrix} \xi_{i+1} \\ \eta_{i+1} \end{bmatrix} = \begin{bmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{bmatrix} \begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix}$$ i.e. rotation in neighbourhood of (0,0).

Can express any vector as a linear combination of ξ_i and η_i (and make that combination real)

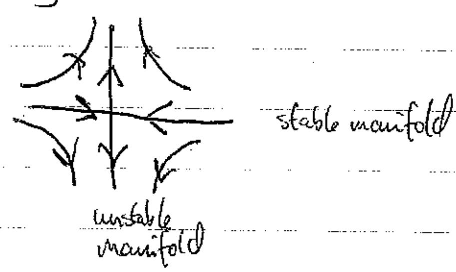
Fixed pt is called a stable or elliptic pt



(circles in ξ, η coords)

2) $$\begin{bmatrix} \xi_{i+1} \\ \eta_{i+1} \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \frac{1}{\lambda} \end{bmatrix} \begin{bmatrix} \xi_i \\ \eta_i \end{bmatrix}$$ hyperbolic motion in neighbourhood of (0,0)

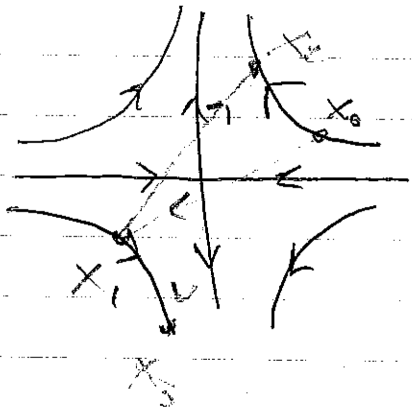
a) $\lambda > 0$ regular hyperbolic fixed pt (iterates stay on same branch)



b) $\lambda < 0$ hyperbolic with reflection fixed pt. Successive iterates jump backwards and forwards between opposite branches

i.e. $\begin{bmatrix} \delta\xi_1 \\ \delta\eta_1 \end{bmatrix} = \begin{bmatrix} -|\lambda|\,\delta\xi_0 \\ -\frac{1}{|\lambda|}\,\delta\eta_0 \end{bmatrix}$, $\begin{bmatrix} \delta\xi_2 \\ \delta\eta_2 \end{bmatrix} = \begin{bmatrix} |\lambda|^2\,\delta\xi_0 \\ \frac{1}{|\lambda|^2}\,\delta\eta_0 \end{bmatrix}$, $\begin{bmatrix} \delta\xi_3 \\ \delta\eta_3 \end{bmatrix} = \begin{bmatrix} -|\lambda|^3\,\delta\xi_0 \\ -\frac{1}{|\lambda|^3}\,\delta\eta_0 \end{bmatrix}$

etc

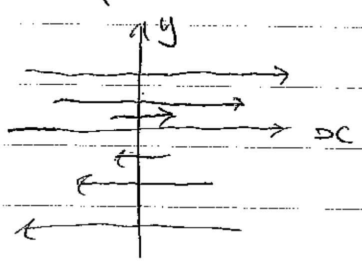


3) $\lambda_+ = \lambda_- = \pm 1$ In original variables $(\delta x_i, \delta y_i)$ we can always write this case as (choosing $\lambda = +1$)

$$\begin{bmatrix} \delta x_{i+1} \\ \delta y_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \delta x_i \\ \delta y_i \end{bmatrix}$$

i.e. $\det T = 1$ as req'd.

where c is any constant. This corresponds to translation parallel to x-axis.



parabolic fixed pt.