Charged particle in an EM field

Lorentz force $\vec{F} = q(\vec{E} + \vec{v} \wedge \vec{B})$

where $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$ $\qquad \vec{B} = \vec{\nabla} \wedge \vec{A}$

$\phi(\vec{r},t)$ = scalar potential
$\vec{A}(\vec{r},t)$ = vector potential

What Lagrangian gives the Lorentz force?

$$\boxed{L = \frac{1}{2} m \dot{\vec{r}}^2 + q\, \dot{\vec{r}} \cdot \vec{A}(r,t) - q\,\phi(\vec{r},t)}$$

Check $\quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) = \frac{\partial L}{\partial q_\alpha}$

check the x-component: $\frac{\partial L}{\partial \dot{x}} = m\dot{x} + q A_x = P_x$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = m\ddot{x} + q\left(\frac{\partial A_x}{\partial t} + \underbrace{\vec{\nabla} A_x \cdot \dot{\vec{r}}}_{\partial_x A_x \dot{x} + \partial_y A_x \dot{y} + \partial_z A_x \dot{z}}\right)$$

A_x varies in time both because of its explicit time-dependence and because of its dependence on particle position $\vec{r}(t)$

RHS: $$\frac{\partial L}{\partial x} = \underbrace{q\,\dot{\vec{r}} \cdot \frac{\partial}{\partial x}\vec{A}}_{\text{i.e. } q\left(\dot{x}\frac{\partial}{\partial x}A_x + \dot{y}\frac{\partial}{\partial x}A_y + \dot{z}\frac{\partial}{\partial x}A_z\right)} - q\frac{\partial \phi}{\partial x}$$

Combine: terms in $\dot{x}$ cancel

$$m\ddot{x} + q\left(\frac{\partial A_x}{\partial t} + \dot{y}\,\partial_y A_x + \dot{z}\,\partial_z A_x\right) = q\left(\dot{y}\frac{\partial}{\partial x}A_y + \dot{z}\frac{\partial}{\partial x}A_z\right) - q\frac{\partial \phi}{\partial x}$$

$$m\ddot{x} = q\Big(\underbrace{-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t}}_{E_x} + \underbrace{\dot{y}\,\partial_x A_y + \dot{z}\,\partial_x A_z - \dot{y}\,\partial_y A_x - \dot{z}\,\partial_z A_x}_{(\vec{V}\wedge\vec{B})_x = \left[\vec{V}\wedge(\vec{\nabla}\wedge\vec{A})\right]_x}\Big)$$

$$\left[V\wedge(\nabla\wedge A)\right]_i$$

$$= \varepsilon_{ijk}\, V_j\, (\nabla\wedge A)_k = \varepsilon_{ijk}\, V_j\, \varepsilon_{klm}\, \partial_l A_m$$

$$= \varepsilon_{ijk}\, \varepsilon_{klm}\, V_j\, \partial_l A_m$$

but $\varepsilon_{ijk}\varepsilon_{klm} = \varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

so $\left[V\wedge(\nabla\wedge A)\right]_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\, V_j\, \partial_l A_m$

put $i=x$ $\Rightarrow l=x$ $\Rightarrow m=x$

$$= V_j\, \partial_x A_j - V_j\, \partial_j A_x$$

$$= V_x\, \partial_x A_x + V_y\, \partial_x A_y + V_z\, \partial_x A_z$$
$$- V_x\, \partial_x A_x - V_y\, \partial_y A_x - V_z\, \partial_z A_x \quad \checkmark$$

Same as

Note ① L contains $\dot{\vec{r}}$ i.e. linear in time derivative.
L cannot be seperated into two parts, $T' - V'$, one quadratic in $\dot{\vec{r}}$ and one independent of it.

② $\vec{p} = m\dot{\vec{r}} + q\vec{A}$ canonical momentum

③ Hamiltonian

$$H = \vec{p}\cdot\dot{\vec{r}} - L = m\dot{\vec{r}}^2 + \cancel{q\dot{\vec{r}}\cdot\vec{A}} - \frac{1}{2}m\dot{\vec{r}}^2 - \cancel{q\dot{\vec{r}}\cdot\vec{A}} + q\phi$$

$$= \frac{1}{2}m\dot{\vec{r}}^2 + q\phi$$

but $\dot{\vec{r}} = \dfrac{\vec{p} - q\vec{A}}{m}$

$$H = \frac{(\vec{p} - q\vec{A})^2}{2m} + q\phi$$

If $\vec{E}$ and $\vec{B}$ are time independent, can choose ϕ and $\vec{A}$ to also be time independent

$\Rightarrow H = E = \text{const}$

Phase space

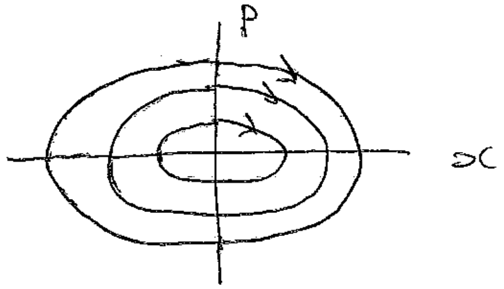
Hamilton's approach to classical mechanics lives in a $2n$ dimensional space called <u>phase space</u> : (x, p)

<u>Example</u> simple harmonic oscillator

$$H = \frac{p^2}{2m} + \frac{1}{2} k x^2 = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$$

$$\text{where } \omega = \sqrt{\frac{k}{m}}$$

Contours of constant energy $H = E$ are ellipses



phase space

Hamilton's eqns:
$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$
$$\dot{p} = -\frac{\partial H}{\partial x} = -m\omega^2 x$$

Liouville's theorem

General motion in phase space might be complicated, but according to Hamilton's equations it is divergenceless, i.e. they generate "incompressible flow".

Divergence : $\nabla . V = \partial_x V_x + \partial_y V_y$ (in 2D)

In phase space : $\sum_{i=1}^{n} \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right)$

$$= \sum_{i=1}^{n} \left(\frac{\partial^2 H}{\partial q_i \partial p_i} - \frac{\partial^2 H}{\partial p_i \partial q_i} \right) = 0$$

A blob of phase space "fluid" has zero divergence.
A ~~Volume~~ Volume element in phase space is preserved under Hamiltonian fluid.
Lagrangian ~~motion~~ dynamics do not conserve volume in $(x, \dot{x})$ "space".

In Hamiltonian dynamics we consider p and q on equal footing as a single set of 2n coords Z.
$\vec{Z} = (q_1, q_2 \ldots q_n ; p_1, p_2 \ldots p_n)$

then Hamilton's equations: $\dot{\vec{Z}} = J . \nabla H(z)$

$$\nabla = (\partial z_1, \partial z_2, \ldots \partial z_n)$$

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$ 2n x 2n matrix "The symplectic matrix"

$\mathbb{1}$ = n×n unit matrix

Poisson brackets

Time-dependence of dynamical quantities can be formulated elegantly in the Hamiltonian picture

consider $f = f(p, q, t)$ (some function)

$$\frac{df}{dt} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial f}{\partial p_i}\frac{dp_i}{dt} \right) + \frac{\partial f}{\partial t}$$

$$= \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t}$$

$$= [f, H]_{q,p} + \frac{\partial f}{\partial t}$$

N.B. writing it as $[H, f]$ is a convention Could equally write $[f, H]$

$[f, H]$ = the Poisson bracket of f with H.

There is a close analogy to ~~Q.M.~~ commutators in Q.M.

Can write the Poisson bracket for any pair of dynamical quantities

$$[f, g]_{q,p} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

If a quantity is explicitly time independent $(f = f(p, q))$ and its Poisson bracket with H vanishes, then f is a constant of the motion.

Poisson bracket of H with itself is zero. So energy of a time-independent system $(H = E)$ is a constant of the motion

Compare with the Heisenberg equations of motion: (actually first written down by Dirac)

$$i\hbar \dot{\hat{A}} = [\hat{A}, \hat{H}]$$ ← commutator

Note that for three functions f, g, h

$$[f, g] = -[g, f]$$

$$[f+g, h] = [f, h] + [g, h]$$

$$[fg, h] = f[g, h] + g[f, h]$$

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0 \qquad \text{(Jacobi's identity)}$$

These properties show that Poisson brackets satisfy a <u>Lie Algebra</u>
These same identities are satisfied by Poisson brackets and commutators.

If we choose f, g, h to be canonical variables

$$[q_a, q_b] = \sum_{i=1}^{n} \left(\frac{\partial q_a}{\partial q_i} \underbrace{\frac{\partial q_b}{\partial p_i}}_{\text{always zero}} - \underbrace{\frac{\partial q_a}{\partial p_i}}_{\text{always zero}} \frac{\partial q_b}{\partial q_i} \right) = 0$$

$$[p_a, p_b] = 0$$

$$[q_a, p_b] = \sum_{i=1}^{n} \left(\frac{\partial q_a}{\partial q_i} \frac{\partial p_b}{\partial p_i} - \underbrace{\frac{\partial q_a}{\partial p_i}}_{=0} \underbrace{\frac{\partial p_b}{\partial q_i}}_{=0} \right) = \delta_{ab}$$

cf QM $\qquad [\hat{q}_a, \hat{q}_b] = i\hbar\, \delta_{ab}$

Dirac's rule for quantization $\quad [\,,\,]_{\text{classical}} \to \dfrac{[\,,\,]_{\text{quantum}}}{i\hbar}$

There are cases where the derivatives appear in q.m. version, e.g. the very useful relations

$$[\hat{x}_i, F(\hat{p})] = i\hbar \frac{\partial F}{\partial p_i}$$

$$[\hat{p}_i, G(x)] = -i\hbar \frac{\partial G}{\partial x_i}$$

Can be proved using $[A,BC] = [A,B]C + B[A,C]$ repeatedly

Classical proof

Take first one : $[x, F(p)] = \frac{\partial x}{\partial x}\frac{\partial F}{\partial p} - \underbrace{\frac{\partial x}{\partial p}}_{=0}\underbrace{\frac{\partial F}{\partial x}}_{=0}$

Now apply $[\ ,\ ]_{classical} \Rightarrow \frac{[\ ,\ ]_{quantum}}{i\hbar}$

$$[x, F(p)] \rightarrow \frac{[\hat{x}, F(\hat{p})]}{i\hbar} = \frac{\partial F}{\partial p}$$

as req'd.

Note: Quantum commutators also work for cases when there is no classical analogue, e.g. spin