Chaos

The surface of section (also called the Poincaré return map)

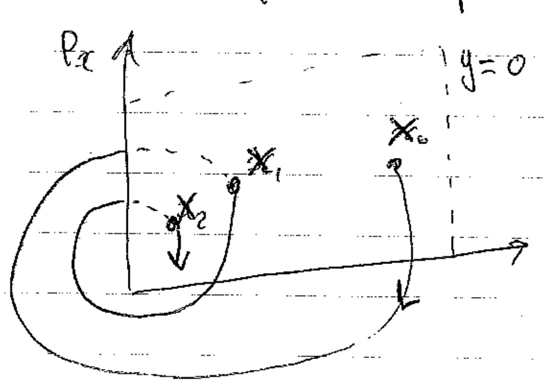
How do we detect chaos? For integrable systems Motion on tori is ordered. Slightly different initial conditions lead to trajectories on different (but close) tori with slightly different frequencies $\Rightarrow$ distance between trajectories increases linearly with time.

If the tori don't exist then motion is less constrained. These two different scenarios can be diagnosed by the <u>surface of section</u> due to Poincaré (1892) and Birkhoff (1932). For chaos in an autonomous system we need $n \geq 2$, i.e. at least 4 dim phase space $\Rightarrow$ hard! Idea: visualize using a 2D "stroboscope"

e.g. ~~take~~ $n=2$ $\qquad H = \frac{1}{2m}(p_x^2+p_y^2) + V(x,y)$

On a given energy shell take a slice of phase space at some point e.g. $y=0$. Now follow orbit (e.g. integrate Hamilton's equations on a computer) and every time it passes through $y=0$ we record p_x and x.

If motion is bounded, the orbit will repeatedly pass through this slice and build up a map of successive (p_x, x) values



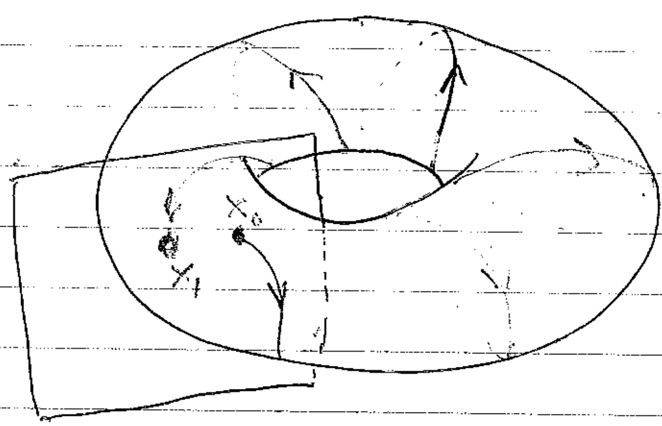
trajectory
$(X_0, X_1, X_2, \ldots)$
$\uparrow$
(p_x, x)

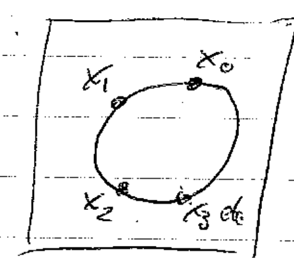
These values define state of system to within a sign: given E and $y=0$

$$P_y = \pm \sqrt{2m\left(E - \tfrac{1}{2m}P_x^2 - V(x,0)\right)}$$

Normally we just keep one sign of P_y, e.g. $P_y > 0$.
For simplicity consider $n=2$.
If motion lies on a torus then there are two possibilities:

1) Frequencies are rational $\frac{\omega_1}{\omega_2} = \frac{m}{n}$ then orbit is closed and there are only a finite number of intersections $X_0 = X_n$





2) Frequencies are irrational : orbit is quasi-periodic and a single orbit covers torus ergodically. Surface of section is a smooth curve which gradually fills up with iterates X_i

If motion is not confined to a torus then we get a random splatter of points.

According to KAM thm phase space can be <u>mixed</u> i.e. some areas of chaos and some areas which are regular.

See Power point slides for Poincare sections of the Henon - Heiles system (up to Dicke model).

<u>Henon - Heiles Hamiltonian</u>

$$H = \frac{1}{2}\left(P_x^2 + P_y^2 + x^2 + y^2\right) + x^2 y - \frac{1}{3} y^3$$

- originally chosen as a simple model for a star in a cylindrically symmetric gravitationally smoothed galactic potential (also provides a simple model for nonlinear molecular bonds.

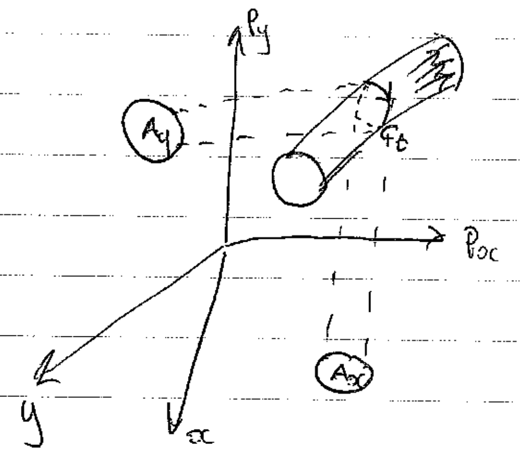
- for small displacements the motion is nearly linear
- big displacements (higher energy) sample the nonlinearity of potential
- self intersecting curves (at A, B, C) appear smooth at intersections but aren't when you zoom in.
- at $E = \frac{1}{8}$ there is a chain of 5 islands (generated from a single trajectory that jumps from island to island.
- random splatter of pts is also generated by 1 trajectory (at $E = \frac{1}{8}$.)
- $E = \frac{1}{6}$ made from a single trajectory (predominantly "chaotic" sea")

Can consider the Poincaré section as a symplectic mapping:

Consider a tube of trajectories in $2n$-dim phase space encircled by a closed curve C

$$\oint_C \vec{p}\cdot d\vec{q} = \oint_{C_t} \vec{p}\cdot d\vec{q}$$

N.B. this corresponds to sum of areas projected onto the set of (p_i, q_i) planes



e.g. $n=2$

$A_x + A_y$ ~~is~~ $=$ const.

Proof that mapping is symplectic is similar to earlier proof for canonical transformations.

For bounded motion the tube will eventually pass back through surface at $y=0$. However there is no reason for each point to pass through $y=0$ at the same time $\Rightarrow$ must consider Poincaré-Cartan invariant

$p\,dq - H\,dt$ is invariant under canonical transformations cf. $\int (p\dot{q} - H)\,dt = \int L\,dt = S$

$$\oint_C p_x\,dx + p_y\,dy - H\,dt = \oint_{C_t} p_x\,dx + p_y\,dy - H\,dt$$

However when $E = H = \text{const}$ $\quad \underbrace{\oint_C H\,dt = \oint_{C_t} H\,dt = 0}$

H is const so integral around a closed curve gives zero.

Also $y=0$ so $\oint_C p_y\,dy = \oint_{C'} p_y\,dy = 0$

so $\oint_C p_x\,dx = \oint_{C'} p_x\,dx$

Area preserving mappings

Poincaré sections motivate the study of maps. These can contain ~~the~~ all the features of continuous dynamical systems but are easier to study (especially numerically)

The twist map

Consider an integrable, isoenergetic, non-degenerate sytem
$\Rightarrow$ nested tori with a frequency ratio which varies smoothly from torus to torus.

Actions : (I_1, I_2) Energy shell $E = H(I_1, I_2)$

$\Rightarrow$ linear flow on torus

$$\theta_1(t) = \omega_1 t + \theta_1(0)$$
$$\theta_2(t) = \omega_2 t + \theta_2(0)$$

$\omega_1 = \omega_1(I_1, I_2) = \dfrac{\partial H}{\partial I_1}$ $\qquad$ $\omega_2 = \omega_2(I_1, I_2) = \dfrac{\partial H}{\partial I_2}$

Period for ~~a~~ 2π cycle of θ_2 is $t_2 = \dfrac{2\pi}{\omega_2}$

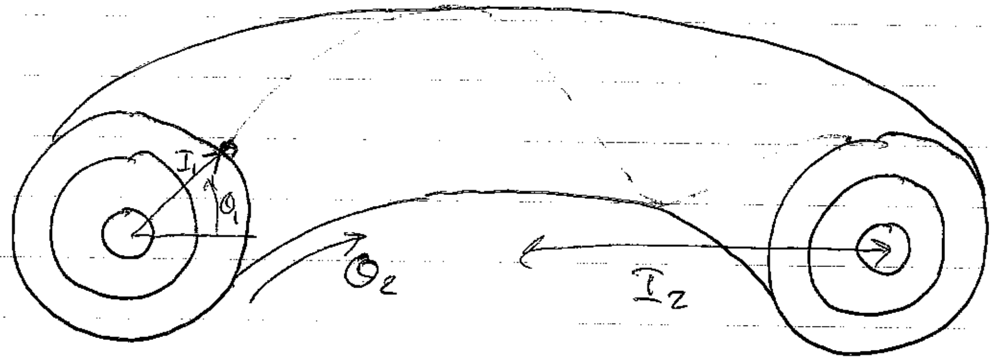
Change in θ_1:

$$\theta_1(t+t_2) = \theta_1(t) + \omega_1 t_2$$
$$= \theta_1(t) + 2\pi \frac{\omega_1}{\omega_2}$$
$$= \theta_1(t) + 2\pi\alpha(I_1)$$

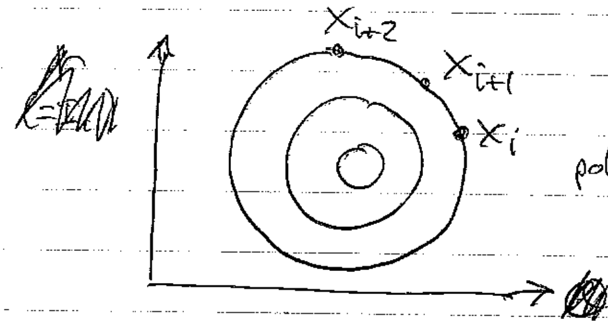
rotation number $\alpha = \dfrac{\omega_1}{\omega_2}$

Can be written as just a function of I_1 because $I_2 = I_2(E, I_1)$

Can now consider (I_1, θ_1) plane as a Poincaré section:



$\Rightarrow$



polar coords: (r, θ)

$$x_i = \left(\theta_1 (t + i t_2), I_1 \right)$$

Define $\left.\begin{array}{l} \theta_i = \theta_1 (t + i t_2) \\ r = I_1 \end{array}\right\} \; x_i = x_i(r, \theta)$

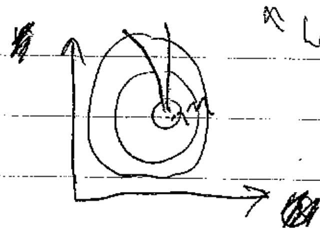
Map for flow on a torus with a given radius I_1

$$T : \quad \begin{aligned} \theta_{i+1} &= \theta_i + 2\pi \alpha (r_i) \\ r_{i+1} &= r_i \end{aligned}$$

T maps pts around a circle, uniformly for irrational α and discretely for rational α.

~~However~~ A radial line of pts will be twisted under T (because $\alpha(r)$ increases with r)

$\wedge$ hence name.



T preserves area:

$$\frac{\partial(\theta_{i+1}, r_{i+1})}{\partial(\theta_i, r_i)} = \begin{vmatrix} \frac{\partial \theta_{i+1}}{\partial \theta_i} & \frac{\partial \theta_{i+1}}{\partial r_i} \\ \frac{\partial r_{i+1}}{\partial \theta_i} & \frac{\partial r_{i+1}}{\partial r_i} \end{vmatrix} = \begin{vmatrix} 1 & 2\pi \frac{\partial \alpha}{\partial r_i} \\ 0 & 1 \end{vmatrix}$$

$$= 1$$

N.B. doesn't matter if we write α as fn of r_i or r_{i+1}.

To make non-integrable we can add a nonintegrable perturbation

$$T_\epsilon : \quad \theta_{i+1} = \theta_i + 2\pi\alpha(r_i) + \epsilon f(r_i, \theta_i)$$

$$r_{i+1} = r_i + \epsilon\, g(r_i, \theta_i)$$

f and g are chosen to preserve area of mapping (Actually Moser's contribution in 1962 to KAM thm was to show that for sufficiently small ϵ circles with sufficiently irrational winding numbers are preserved.)

<u>Henon map</u>

Consider $\quad T: \quad x_{i+1} = f(x_i, y_i)$

$$y_{i+1} = g(x_i, y_i)$$

area preserving if $\dfrac{\partial(x_{i+1}, y_{i+1})}{\partial(x_i, y_i)} = 1$.

If f and g are polynomials, mapping is an <u>entire Cremona transformation</u>.

~~A~~ linear examples

1)

$$T: \quad \left.\begin{aligned} x_{i+1} &= x_i \cos\alpha - y_i \sin\alpha \\ y_{i+1} &= x_i \sin\alpha + y_i \cos\alpha \end{aligned}\right\} \text{ rotation by angle } \alpha.$$

2)

$$T: \quad \left.\begin{aligned} x_{i+1} &= x_i + y_i \\ y_{i+1} &= y_i \end{aligned}\right\} \text{ linear shear parallel to } x\text{-axis}$$

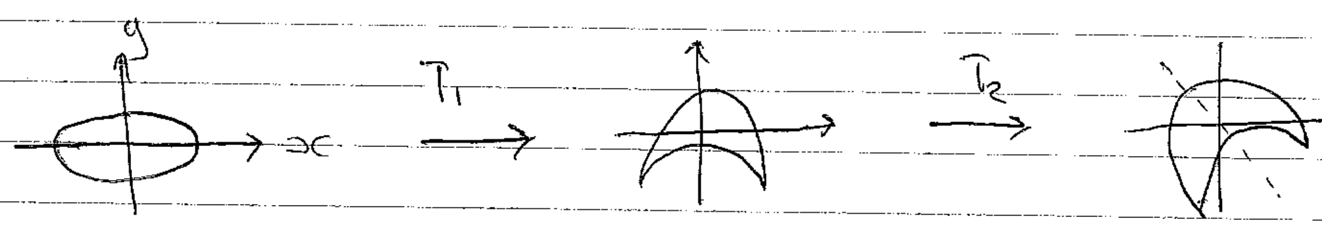
Henon map (non linear)

$$T: \quad \begin{aligned} x_{i+1} &= x_i \cos\alpha - (y_i - x_i^2)\sin\alpha \\ y_{i+1} &= x_i \sin\alpha + (y - x_i^2) \cos\alpha \end{aligned}$$

Can be decomposed into $T = T_1 T_2$

$$T_1: \quad \left.\begin{aligned} x_{i+\frac{1}{2}} &= x_i \\ y_{i+\frac{1}{2}} &= y_i - x_i^2 \end{aligned}\right\} \text{ non-linear shear}$$

$$T_2: \quad \left.\begin{aligned} x_{i+1} &= x_{i+\frac{1}{2}} \cos\alpha - y_{i+\frac{1}{2}} \sin\alpha \\ y_{i+1} &= x_{i+\frac{1}{2}} \sin\alpha + y_{i+\frac{1}{2}} \cos\alpha. \end{aligned}\right\} \text{ simple rotation}$$



Note that mapping is invertible (time reversible)

$$T^{-1}: \quad x_i = x_{i+1}\cos\alpha + y_{i+1}\sin\alpha$$
$$y_i = -x_{i+1}\sin\alpha + y_{i+1}\cos\alpha + (x_{i+1}\cos\alpha + y_{i+1}\sin\alpha)^2$$

Numerical studies of Henon's map show same typical features as the Henon-Heiles Poincaré section: smooth curves, island chains and chaotic splatter

Note that the separtix-like structure looks smooth on one scale but when enlarged we find rich fine structure of island chains in a sea of chaos

[Show slides of Henon map and Logistic map : "Poincaré Section.ppt"]

Area preserving maps and their connection to Hamiltonians (Didn't cover pp 9-10 in class)

Can area preserving maps be derived explicitly from Hamiltonians?

e.g. $H = \frac{1}{2}p^2 + V(q)$

Hamilton's eqns
$$\dot{q} = p$$
$$\dot{p} = -\frac{\partial V}{\partial q}$$

Write $\dot{q} = \frac{(q_{i+1} - q_i)}{\Delta t}$ where $q_{i+1} = q(t+\Delta t)$
$q_i = q(t)$

$\Rightarrow$ H's eqns

$$\left.\begin{aligned} q_{i+1} &= q_i + p_i \Delta t \\ p_{i+1} &= p_i - \Delta t \left(\frac{\partial V}{\partial q_i}\right)_{q=q_i} \end{aligned}\right\} \text{ Not area preserving!}$$

$$\frac{\partial(q_{i+1}, p_{i+1})}{\partial(q_i, p_i)} = \begin{vmatrix} \frac{\partial q_{i+1}}{\partial q_i} & \frac{\partial q_{i+1}}{\partial p_i} \\ \frac{\partial p_{i+1}}{\partial q_i} & \frac{\partial p_{i+1}}{\partial p_i} \end{vmatrix} = \begin{vmatrix} 1 & \Delta t \\ -\Delta t \frac{\partial^2 V}{\partial q_i^2} & 1 \end{vmatrix}$$

$$= 1 + (\Delta t)^2 \frac{\partial^2 V}{\partial q_i^2}$$

Δt is <u>finite</u> for a map.

~~Now~~ Can make area preserving by changing to:

$$p_{i+1} = p_i - \Delta t \left(\frac{\partial V}{\partial q}\right)_{q = q_{i+1}}$$

What sort of Hamiltonian would give rise to <u>precisely</u> these equations of motion? (i.e. not Poincaré map)

<u>Ans</u>: time-dependent Hamiltonian

$$H(p,q,t) = \begin{cases} \frac{p^2}{2\gamma} & 0 < t < \gamma T \\ \frac{V(q)}{1-\gamma} & \gamma T < t < T \end{cases} \quad 0 < \gamma < 1$$

i.e. free propagation for time γT followed by impulsive force due to potential $V(q)$ for time $(1-\gamma)T$
e.g. ray propagation through periodically spaced lenses.

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{\gamma}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -\frac{\partial V}{\partial q}\frac{1}{1-\gamma}$$

Integrate over time $0 \rightarrow T$ i.e. one period of $t = iT$ to $t = (i+1)T$

$$q_{i+1} = q_i + \frac{p}{\gamma}\,\gamma T$$

$$p_{i+1} = p_i - \left.\frac{\partial V}{\partial q}\right|_{q=q_{i+1}} \frac{1}{1-\gamma}(1-\gamma)T$$

same as first set of eqns as above if we associate $T \rightarrow \Delta t$