Canonical transformations

In Lagrangian mechanics (generalized coords q_i and velocities $\dot{q}_i$) it was sometimes convenient to transform to new ~~variables~~ coords

$$Q_i = Q_i(q_1, \dots q_n)$$

In the Hamiltonian description p and q are on equal footing, so we consider transformations from one set of phase space variables (p_i, q_i) to a new set (P_i, Q_i)

$$P_i = P_i(q_1, \dots q_n; p_1, \dots p_n)$$
$$Q_i = Q_i(q_1, \dots q_n; p_1, \dots p_n)$$

These transformations are called CANONICAL TRANSFORMATIONS if the form of Hamilton's equations is preserved

$$\dot{Q}_i = \frac{\partial H'(Q,P)}{\partial P_i} \qquad \dot{P}_i = -\frac{\partial H'}{\partial Q_i}(Q,P)$$

$H' = H'(Q,P)$ is the transformed Hamiltonian

Symplectic structure is preserved.

Preservation of phase volume

CTs preserve phase volume

$$\int \prod_{i=1}^{n} dp_i\, dq_i = \int \prod_{i=1}^{n} dP_i\, dQ_i$$

integral over a prescribed volume in phase space

In general the two integrals are related by the Jacobian of the transformation

$$\int \prod_{i=1}^{n} dP_i\, dQ_i = \int \frac{\partial(P_1, \ldots, P_n ; Q_1, \ldots Q_n)}{\partial(p_1, \ldots, p_n ; q_1, \ldots q_n)} \prod_{i=1}^{n} dp_i\, dq_i$$

Volume preserving transformation has unit Jacobian

$$\frac{\partial(P_1, \ldots P_n ; Q_1 \ldots Q_n)}{\partial(p_1, \ldots p_n ; q_1, \ldots, q_n)} = \frac{\partial(p_1, \ldots p_n ; q_1, \ldots q_n)}{\partial(P_1, \ldots P_n ; Q_1, \ldots Q_n)} = 1$$

Example 1) $Q = -p \quad , \quad P = q$

$$\frac{\partial(P,Q)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial P}{\partial p} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial Q}{\partial q} \end{vmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$$

$\Rightarrow$ Canonical (volume preserving)

N.B. this indicates on how equal a footing p and q are. They can be interchanged, but with a sign change (cf Hamilton's eqns)

Example 2) $q = P \cos Q \quad , \quad p = P \sin Q$ (transfn from polar $\to$ cartesian)

$$\frac{\partial(q,p)}{\partial(Q,P)} = \begin{vmatrix} \frac{\partial q}{\partial Q} & \frac{\partial p}{\partial Q} \\ \frac{\partial q}{\partial P} & \frac{\partial p}{\partial P} \end{vmatrix} = \begin{vmatrix} -P\sin Q & P\cos Q \\ \cos Q & \sin Q \end{vmatrix} = -P$$

$\Rightarrow$ not canonical (not volume preserving)

Liouville's thm says that phase volume is preserved under Hamiltonian flow
$\Rightarrow$ can view Hamiltonian flow as a canonical transformation

Consider phase space trajectory $q_0, p_0 \rightarrow q_1, p_1$ in short time δt

$$q_1 = q(t+\delta t) = q_0 + \delta t \left.\frac{dq}{dt}\right|_{t=t_0} + O(\delta t^2)$$

$$= q_0 + \delta t \frac{\partial}{\partial p_0} H(q_0, p_0, t) + O(\delta t^2)$$

$$p_1 = p(t+\delta t) = p_0 + \delta t \left.\frac{dp}{dt}\right|_{t=t_0} + O(\delta t^2)$$

$$= p_0 - \delta t \frac{\partial}{\partial q_0} H(q_0, p_0, t) + O(\delta t^2)$$

If transformation $q_0, p_0 \rightarrow q_1, p_1$ is canonical then Jacobian must be unity:

$$\frac{\partial(q_1, p_1)}{\partial(q_0, p_0)} = \begin{vmatrix} \frac{\partial q_1}{\partial q_0} & \frac{\partial p_1}{\partial q_0} \\ \frac{\partial q_1}{\partial p_0} & \frac{\partial p_1}{\partial p_0} \end{vmatrix} = \begin{vmatrix} 1 + \delta t \frac{\partial^2 H}{\partial q_0 \partial p_0} & -\delta t \frac{\partial^2 H}{\partial q_0^2} \\ \delta t \frac{\partial^2 H}{\partial p_0^2} & 1 - \delta t \frac{\partial^2 H}{\partial q_0 \partial p_0} \end{vmatrix}$$

$$= 1 + O(\delta t^2)$$

$$= 1 \quad \text{in lim } \delta t \rightarrow 0$$

Phase space volume is preserved $\Rightarrow$ canonical

The Optimal transformation

<u>Qu</u>: Which transformation makes the integration of Hamilton's eqns as simple as possible?

<u>Ans</u>: ~~New~~ ~~the~~ H' depends only on P_i (new momenta)

$$H(p_1, \ldots p_n, q_1, \ldots q_n) \longrightarrow H'(P_1, \ldots P_n)$$

Then
$$\dot{P}_i = -\frac{\partial H'}{\partial Q_i} = 0 \quad \Rightarrow P_i = \text{const} \quad i = 1, \ldots n$$

$$\dot{Q}_i = \frac{\partial H'}{\partial P_i} = f_i(P_1, \ldots P_n)$$

time-independent fn of the P_i

Can immediately integrate 2nd eqn

$$Q_i = f_i \, t + \delta_i \qquad i = 1, \ldots n$$

arbitrary consts set by initial conditions

* Q_i are now "cyclic" coords.

We have achieved a complete integration of eqns of motion

P_i and δ_i form a complete set of $2n$ integrals

P_i = n non-trivial constants of motion ("first integrals")

δ_i = n trivial consts of integration (needed to complete the integration)

Need to know : 1) how to find these magical new variables
2) how to correctly transform Hamiltonian into new representation

Generating functions

Canonical transformations are effected by generating functions.
Can find by variational principle. Simpler route (for time-independent problems) is via phase-volume preservation.

For simplicity consider 1 degree of freedom

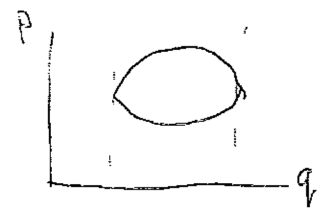
$$\iint_R dp\,dq = \iint_R dP\,dQ$$

some region

Stokes thm : $\iint dA = \oint dl$ line integral

$$\oint_C p\,dq = \oint_C P\,dQ$$

encloses R



Area enclosed : $\iint dp\,dq$

and also $\int_{\text{upper curve}} p\,dq - \int_{\text{lower curve}} p\,dq = \oint p\,dq$ [clockwise]

Now $P = P(p,q)$, $Q = Q(p,q)$

but could also write as $P = P(Q,q)$, $p = p(Q,q)$

i.e. Q and q are now considered as independent variables

$$\oint_C [p(Q,q)\,dq - P(Q,q)\,dQ] = 0$$

i.e. doesn't depend on path, only on position in plane.

Implies integrand must be the total differential of ~~some~~ some function, call it $F_1(Q,q)$

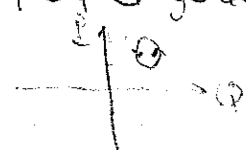
$$\oint_C [p\,dq - P\,dQ] = \oint_C dF_1(Q,q) = \oint_C \left(\frac{\partial F_1}{\partial Q}dQ + \frac{\partial F_1}{\partial q}dq\right)$$

then
$$\underbrace{p = \frac{\partial}{\partial q}F_1(Q,q)}_{}\ , \quad P = -\frac{\partial}{\partial Q}F_1(Q,q)$$

establishes relationship between p and (q,Q)
→ invert to find $\boxed{Q(q,p)}$ $\xrightarrow{\text{sub into 2nd eqn}}$ → to give other desired relationship $P(q,p)$
(need $\frac{\partial^2 F_1}{\partial q\,\partial Q} \neq 0$)

For F_1 we chose (q,Q) as independent variables.
Can also choose, e.g. (P,q) , (Q,p) , (P,p)

Consider the (P,q) combination : ~~use $\oint d(pQ) = \oint p\,dQ + \oint Q\,dp$~~



$\oint P\,dQ$ anticlockwise $= -\oint Q\,dP$ clockwise

sub in ~~$\oint[p\,dq - P\,dQ] = 0$ again~~

$$\iint_R dp\,dq = \iint_R dP\,dQ \quad \Rightarrow \quad \oint_C p\,dq = -\oint_C Q\,dP$$

[Alternatively $\oint d(PQ) = 0 = \oint P\,dQ + \oint Q\,dP$

$$\oint(p\,dq + Q\,dP) = 0 = \oint dF_2(P,q) = \oint\left[\frac{\partial F_2}{\partial P}dP + \frac{\partial F_2}{\partial q}dq\right]$$

So $$p = \frac{\partial F_2}{\partial q}(P,q) \qquad Q = \frac{\partial}{\partial P} F_2(P,q)$$

Example: $F_2(P,q) = Pq$

$$p = \frac{\partial F_2}{\partial q} = P \qquad Q = \frac{\partial F_2}{\partial P} = q \quad \Rightarrow \text{identity transform!}$$

There are also two other generating functions: $F_3(Q,p)$ and $F_4(P,p)$ but $F_2(P,q)$ is the most important

If the canonical transformation is time independent, the transformation from $H(p,q)$ to $H'(P,Q)$ is a straight forward change of variables

$$H' = H'(P,Q) = H(p(P,Q),\, q(P,Q))$$

Can easily show using chain rule and $\frac{\partial(P,Q)}{\partial(p,q)} = 1$ that

$$\dot{Q} = \frac{\partial H'}{\partial P} = \frac{\partial H}{\partial P}$$

$$\dot{P} = -\frac{\partial H'}{\partial Q} = -\frac{\partial H}{\partial Q}$$

i.e. that the symplectic form of Hamilton's eqns is preserved in the new variables

Demonstration:

e.g. $$\frac{\partial H'}{\partial P} = \dot{Q} = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial p}\dot{p} = \frac{\partial Q}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial H}{\partial q}$$

$$= \frac{\partial Q}{\partial q}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}\right) - \frac{\partial Q}{\partial p}\left(\frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}\right)$$

$$= \frac{\partial H}{\partial P}\underbrace{\left(\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}\right)}_{=1 \text{ by Jacobian}} = \frac{\partial H}{\partial P}$$

Similarly for other equation $\dot{P} = -\frac{\partial H}{\partial Q}$

In the case of an explicitly time-dependent transformation the above arguments become more involved → can use a variational principle in phase space to show

$$H'(Q,P,T) = H(q,p,t) + \frac{\partial F_i}{\partial t}$$