1995 CAP University Prize Examination

Wednesday, February 1, 1995
2:00 - 5:00 p.m.

Instructions

1. The use of calculators is allowed.

2. Do as many questions as you can, in whole or in part. It is not expected that you will complete all ten questions!

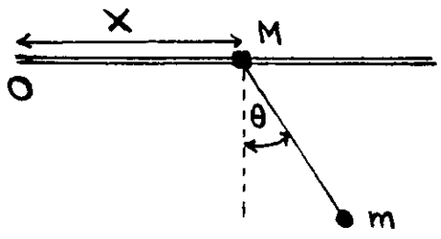
3. Answer each question beginning on a new sheet of paper. Please write the question number and your name at the top of each page.

4. All questions are of equal value, but not of equal difficulty.

---

1. Calculate the mean density of the sun given only the following data: the gravitational constant $G = 6.67 \times 10^{-11}\ Nm^2/kg^2$, the length of the earth's year, $T_E = 3.16 \times 10^7 s$, and the fact that the sun's diameter subtends an angle of 0.55^0 at the earth (i.e. the angular "width" of the sun as seen from the earth is 0.55^0).

2. (a) A simple pendulum, consisting of a bob of mass m attached to a light (massless) rod of length ℓ, swings in a plane, without friction, from a bead of mass M. The bead is free to move without friction on a horizontal wire, as in the diagram. Derive an expression for the Lagrangian L of this system in terms of the generalized coordinates X and θ shown in the diagram.



(b) Write down the Lagrange equations for X and θ, and simplify as much as possible. (For a generalized coordinate q, the Lagrange equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0,$$

where $\dot{q}$ is the time derivative of q.)

(c) From the Lagrange equations, show that if the pendulum's motion started with both M and m at rest (but with θ nonzero and $|\theta| < 90°$),

$$(M+m)\,\dot{X} + m\ell\,\dot{\theta}\cos\theta = 0$$

where X and $\dot{\theta}$ are the time derivatives of X and θ. How may this result be interpreted?

3. A dielectric having a dielectric constant K extends from $-a$ to $+a$ in the x direction and can be considered to have infinite extent in the y and z directions. There are conducting plates on the surfaces at $x = \pm a$. Charge has been implanted in the dielectric and the density of this "free" charge is given by

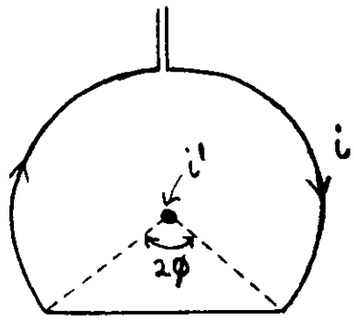
$$\rho(x) = \rho_0 \cosh(\beta x)$$

where ρ_0 and β are positive constants. An external circuit forces the potential difference between the plate at $x = a$ and the plate at $x = -a$ to be V.

(a) Obtain an expression for the electric field in the dielectric in terms of ρ_0, β, K, a and V.

(b) What is the surface charge density of the "free" charge on the conducting plates in this situation? Express your answer in terms of ρ_0, β, K, a and V.

4. A wire follows the circumference of a circle of radius a except for an arc of angular length 2ϕ across which it follows the chord. This loop is suspended from a point opposite the centre of the chord, as shown, so that its plane is perpendicular to a long straight wire that passes through its centre. When the currents are i and i′, show that the torque on the loop is $(\mu_0 i i' a/\pi)\ (\sin\phi - \phi\cos\phi)$. The following indefinite integrals may be helpful:



$$\int \tan^2\theta\, d\theta = \cos\theta\ (\tan\theta - \theta)$$

$$\int \frac{x^2}{x^2+c^2}\, dx = x - c\tan^{-1}(x/c)\,.$$

5. A body of finite mass is originally at a temperature T_1 which is higher than that of a heat reservoir at temperature T_2. Suppose a heat engine operates in a cycle between the body and the reservoir until it lowers the temperature of the body from T_1 to T_2, in the process extracting heat Q from the body. The engine does work W and rejects heat Q-W to the reservoir at T_2. By considering the entropy changes of the body and of the reservoir, show that the maximum amount of work obtainable from the engine is

$$W_{max} = Q - T_2\ (S_1 - S_2)$$

where $S_1 - S_2$ is the entropy decrease of the body.

6. A simple model for solid nitrogen (below 77K) is to treat it as a collection of N independent noninteracting diatomic molecules in thermal equilibrium at temperature T. To calculate the contribution of the rotational motion to the thermodynamic properties of the solid, consider the molecules to be rigid rotors whose centres of mass are stationary at the sites of a crystal lattice. The rotational energy levels of a diatomic molecule have the form,

$$\epsilon(j) = j(j+1)\epsilon_0 \qquad j = 0,1,2,...$$

and the jth level consists of 2j+1 degenerate states.

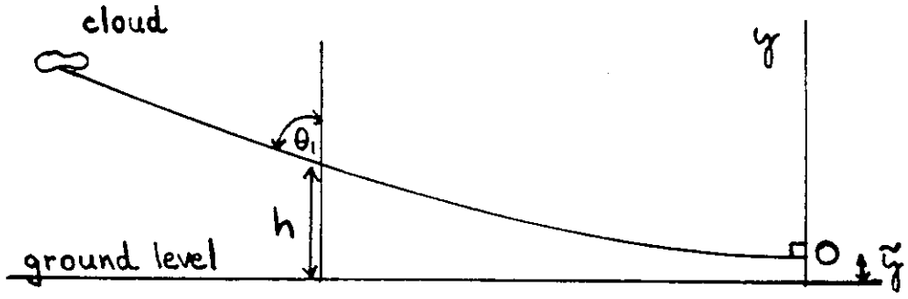
(a) Write down the partition function Z_{rot} for the system of N molecules.

(b) Evaluate the partition function approximately by assuming that j is a continuous variable and converting the sum to an integral.

(c) Calculate the contribution of the rotational motion to the heat capacity at constant volume, C_V. It may be useful to recall that the internal energy U can be obtained from the partition function Z by the relation,

$$U = k_B T^2 \frac{\partial(lnZ)}{\partial T}.$$

7.



A mirage is formed by the bending of light from the sky over the hot surface of the earth, as in the diagram. The index of refraction of the air increases linearly from n_0 at y=0 to n_1 at y=h. Assume that an observer is situated at O in the diagram. For the light ray shown in the diagram,

(a) Determine the height ỹ below which no image of the cloud will be seen by an observer situated on the y-axis.

(b) For what angle θ_1 will the cloud be seen as straight ahead by an observer on the ground (y=0)?

8. A quantum system can exist in two states, $|a_0\rangle$ and $|a_1\rangle$, which are normalised eigenstates of an operator $\hat{A}$ (an observable) with eigenvalues 0 and 1 respectively.

$$\hat{A}|a_0\rangle = 0, \qquad \hat{A}|a_1\rangle = |a_1\rangle.$$

A Hamiltonian $\hat{H}$ for the system is defined by

$$\hat{H}|a_0\rangle = \alpha\hbar|a_0\rangle + \beta\hbar|a_1\rangle,$$

$$\hat{H}|a_1\rangle = \beta\hbar|a_0\rangle + \alpha\hbar|a_1\rangle,$$

where α, β are real constants.

(a) Find the eigenvalues of $\hat{H}$, and the corresponding eigenstates.

(b) The system is in the state $|a_0\rangle$ at time t=0. Find the state of the system at a later time t.

(c) A measurement of the observable A is made at time $t=\pi\hbar/2\beta$. What are the probabilities of obtaining the values 0 and 1 in this measurement?

9. The supernova 1987A is located about 170,000 light years away from the earth. The interactions of a burst of 10 neutrinos from the supernova were observed in an underground tank of water within an interval of 2 secs. The neutrino energies varied in the range between 5 to 20 MeV (1 MeV = 10^6 eV). Estimate the upper limit of the neutrino mass from these data. (Assume that the rest mass m_υ of a neutrino is so small that $pc >> m_\upsilon c^2$). Express your answer for $m_\upsilon c^2$ in eV.

10. Explain briefly the physical principles involved in the operation of any FOUR of the following devices:

(i) thermistor

(ii) photomultiplier

(iii) diode rectifier

(iv) Hall effect probe for measuring magnetic fields

(v) transducer for generating ultrasonic waves.